Modélisation mathématique des courants de bord ouest en océanographie

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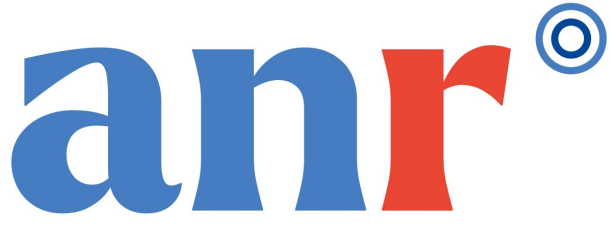
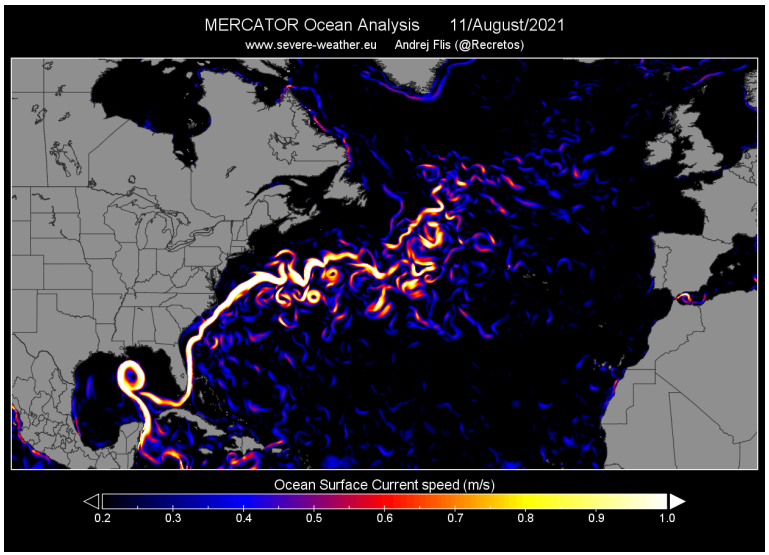


What are WBC anyway?

- Strong, narrow currents near the western coasts of oceanic basins;
  Ex: Gulf Stream, Kuroshio, Agulhas, ...
- Crucial in large scale ocean dynamics: carry heat from equator to the poles;
- Due to variations of Coriolis factor with latitude (β-plane approx.: $\sin\theta \simeq \sin\theta_0 + (\theta - \theta_0)\cos\theta_0$);
- Dissymetry between East and West, but same behavior in northern/southern hemisphere!



Purpose of the talk

Motivations:

- WBC remain poorly described in many GCM (either nonresolved dynamics or sensitive to parametrization)
- PDE analysis: stimulating and varied mathematical problems!

Goal: review of existing results on western boundary currents.

- Focus on theoretical mathematical analysis;
- 2d models: important literature:
  - Linear/nonlinear regimes
  - Effects of coastline geometry

  Caveat: limited to small bathymetry variations
- 3d model: recent result with Corentin Gentil
  Influence of topography of latitude of Gulf Stream separation

Outline

2d models for western boundary currents

2d linear and quasilinear regimes

2d nonlinear viscous regime

3d linear model (with topography & stratification)

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Starting point: homogeneous models of ocean circulation

Ref: [Pedlosky, Ocean Circulation Theory]

Assumptions: 2d incompressible model, single layer of homogeneous fluid, stationary regime, β-plane approximation.

Vorticity balance in nondimensional form:

$$\epsilon J(\psi, \Delta\psi) + \partial_x \psi = w_{\mathrm{Ek}} - \mu\Delta\psi + \mathfrak{E}\Delta^2\psi, \qquad (1)$$

where:

- ψ= stream function ($(u, v) = \nabla^{\perp}\psi$);
- w_{Ek}: Ekman pumping (wind forcing at the surface);
- $J(f, g) = \partial_x g \partial_y f - \partial_x f \partial_y g$;
- $\epsilon, \mathfrak{E}, \mu$: dimensionless parameters, $\ll 1$.

Boundary conditions: for a simply connected ocean basin Ω (no islands),

$$\psi_{|\partial\Omega} = \partial_n \psi_{|\partial\Omega} = 0.$$

Sverdrup balance: $\partial_x \overline{\psi} = w_{\mathrm{Ek}}$.

$\rightarrow$ Boundary layers in vicinity of $\partial\Omega$.

Different boundary layer sizes

$$\epsilon J(\psi, \Delta\psi) + \partial_x \psi = w_{\mathrm{Ek}} - \mu \Delta\psi + \mathfrak{E}\Delta^2\psi, \qquad \epsilon, \mathfrak{E}, \mu \ll 1. \tag{2}$$

- $\delta_I = \epsilon^{1/2}$ (balance of $J(\psi, \Delta\psi)$ with $\partial_x\psi$): inertial boundary layer;
- $\delta_M = \mathfrak{E}^{1/3}$ (balance of $\Delta^2\psi$ with $\partial_x\psi$): Munk boundary layer;
- $\delta_S = \mu$ (balance of $\Delta\psi$ with $\partial_x\psi$): Stommel boundary layer.

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2d linear and quasilinear regimes

2d nonlinear viscous regime

3d linear model (with topography & stratification)

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- Munk and Stommel layers in an idealized basin
- Taking into account the geometry #1: geostrophic degeneracy
- Taking into account the geometry #2: rough boundaries

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Munk layers

$$\cancel{\epsilon J(\psi, \Delta\psi)} + \partial_x \psi = w_{\mathrm{Ek}} \cancel{- \mu \Delta\psi} + \mathfrak{E} \Delta^2 \psi \quad \text{in } \Omega = (x_W, x_E) \times \mathbf{T},$$
$$\psi_{\partial\Omega} = \partial_n \psi_{\partial\Omega} = 0$$

Strategy: look for $\psi \simeq \overline{\psi} + \psi_W^{BL} + \psi_E^{BL}$, with

$$\psi_i^{BL} = \exp(-\lambda |x - x_i|) A_i(y), \quad \lambda \gg 1, \ \Re(\lambda) > 0.$$

Equation for λ: $\lambda^3 = \eta \mathfrak{E}^{-1}$, with $\eta_W = -1$, $\eta_E = +1$.

$\rightarrow$ East/West dissymetry.

Consequence: two boundary conditions can be lifted on the West boundary, but only one on the East boundary

$\Rightarrow \overline{\psi}_{|x=x_E} = 0$, $A_E = O(\mathfrak{E}^{1/3})$; $A_W^{\pm} = O(1)$.

Western boundary current intensification:

$v = \partial_x \psi = O(\mathfrak{E}^{-1/3})$ in Western BL, $O(1)$ elsewhere.

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Mathematical justification and extensions

- [Desjardins&Grenier, '99]: construction of a solution up to any order; addition of friction term $\mu\Delta\psi$ (Stommel layers);
- [Bresch&Simon, '01]: non-constant (but small) bathymetry;
- [Bresch&Colin,'02]: weak convergence result for general domains (including islands).

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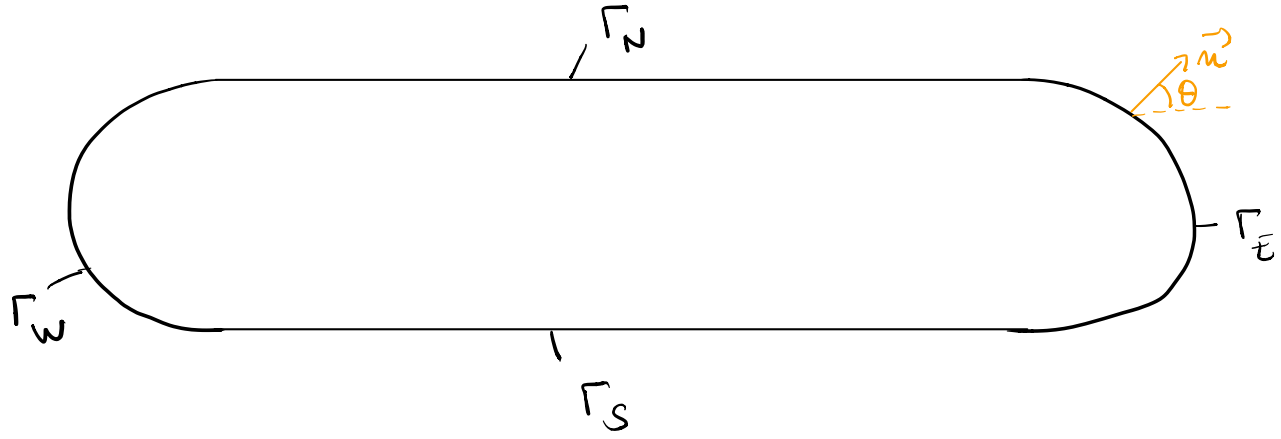
3d linear model (with topography & stratification)

Same equation, different domain

Consider

$$\cancel{\epsilon J(\psi, \Delta\psi)} + \partial_x \psi = w_{\mathrm{Ek}} \cancel{- \mu \Delta \psi} + \mathfrak{E} \Delta^2 \psi, \quad \psi_{\partial\Omega} = \partial_n \psi_{\partial\Omega} = 0,$$

with Ω a smooth, simply connected 2d domain, e.g.



In vicinity of Γ_W, Γ_E, the BL size becomes $(\mathfrak{E}/|\cos\theta|)^{1/3}$: degenerates near Γ_N, Γ_S ($\cos\theta = 0$)

$\rightarrow$ Geostrophic degeneracy.

Degenerate BL near Southern/Western boundaries

Theorem: [D., Saint-Raymond, '18]
In the vicinity of Γ_S, Γ_N, the solution behaves like

$$\overline{\psi} + \psi^{BL}_{N,S}(x, \mathfrak{E}^{-1/4}|y - y_{N,S}|),$$

and $\psi^{BL}_{N,S}$ solves a parabolic equation of the type

$$\partial_x \psi^{BL}_{N,S} - \partial^4_Y \psi^{BL}_{N,S} = 0,$$
$$\psi^{BL}_{N,S|Y=0} = -\overline{\psi}, \quad \partial_Y \psi^{BL}_{N,S|Y=0} = \pm \mathfrak{E}^{1/4} \partial_n \overline{\psi},$$

with initial data at East end of interval.
Difficulty: piece all boundary layers together...
Remark: also allows to treat domains with islands, etc. Similar shear layers in the wake of the island (e.g. Madagascar).
Perspective: nonlinear models?

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Setting of the problem

Consider now the equation

$$\epsilon J(\psi, \Delta\psi) + \partial_x \psi = w_{\mathrm{Ek}} \cancel{-\mu\Delta\psi} + \mathfrak{E}\Delta^2\psi, \quad \psi_{\partial\Omega} = \partial_n \psi_{\partial\Omega} = 0,$$

where $\Omega_{\mathfrak{E}}$ is a domain with rough coasts

$$\Omega_{\mathfrak{E}} := \{(x, y) \in \mathbf{R}\times\mathbf{T}, \quad \mathfrak{E}^{1/3}\gamma_W(y\mathfrak{E}^{-1/3}) \leq x \leq 1 + \mathfrak{E}^{1/3}\gamma_E(y\mathfrak{E}^{-1/3})\}.$$

Remark:

- Roughness chosen so that small scales in x, y have the same size as the linear Munk BL (richest mathematical setting);
- For the same reason, take $\epsilon = \mathfrak{E}$: NL term has the same size as $\partial_x\psi$, $\mathfrak{E}\Delta^2\psi$ in BL;
- Different of (usual) inertial regime because of small scales in y.

Changes in boundary layers

Ansatz becomes

$$\psi_W^{BL} = \Phi_W \left(y; \frac{x}{\mathfrak{C}^{1/3}}, \frac{y}{\mathfrak{C}^{1/3}} \right) = \Phi_W(y, X, Y)$$

(Same for Eastern BL).

Western BL: ODE becomes NL elliptic equation

$$\partial_X \Phi_W + J(\Phi_W, \Delta_{X,Y} \Phi_W) - \Delta_{X,Y}^2 \Phi_W = 0.$$

Eastern BL: ODE becomes linear elliptic equation

$$-\partial_X \Phi_E - \Delta_{X,Y}^2 \Phi_E = 0.$$

Remark: no analytical formula, but analysis remains possible!
[Gérard-Varet & Bresch, '05; Lopez-Ruiz, '21]

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A Prandtl-like equation

Consider now the model

$$\epsilon J(\psi, \Delta\psi) + \partial_x \psi = w_{\mathrm{Ek}} - \cancel{\mu \Delta \psi} + \mathfrak{E} \Delta^2 \psi, \quad \psi_{\partial\Omega} = \partial_n \psi_{\partial\Omega} = 0,$$

with

$$\Omega = \{0 < x < 1, \quad y \in (0, y_0)\}, \quad \epsilon \propto \mathfrak{E}^{2/3}.$$

(In other words, $\delta_I \sim \delta_M$).
The equation in the Western BL now becomes, setting $\Phi = \overline{\psi} + \psi^{BL}$ (total stream function)

$$\partial_X \Phi \partial_y \partial_X^2 \Phi - \partial_y \Phi \partial_X^3 \Phi + \partial_X \Phi - \partial_X^4 \Phi = 0,$$

i.e., after integrating w.r.t. X and setting $(u, v) = \nabla^{\perp}_{X,y} \Phi$,

$$u \partial_X v + v \partial_y v + \Phi - \partial_X^2 v = \overline{\psi}(0, y).$$

$\rightarrow$ Similar to stationary Prandtl equation; tangential variable is now x (unfortunately).

Comments about the Prandtl-like equation

$$
\begin{aligned}
v\partial_y v + u\partial_X v + \Phi - \partial_X^2 v &= \overline{\psi}(0,y),\\
u_{|X=0} = v_{|X=0} &= \Phi_{|X=0} = 0,\\
(u,v) &= \nabla^{\perp}_{X,y}\Phi,\\
\lim_{X\to\infty} \Phi(X,y) &= \overline{\psi}(0,y).
\end{aligned}
\tag{PQG}
$$

- Nonlocal equation;
- $\sim$ Parabolic equation: forward when $v > 0$, backward when $v < 0$;
- The presence of the term Φ breaks the scaling invariance of Prandtl;
- Important source of difficulty: nonlocal transport term $u\partial_X v$ (loss of one derivative).

Remark: focus on stationary setting in this talk. Results in time-dependent setting by [X. Wang & Y.G. Wang, '20].

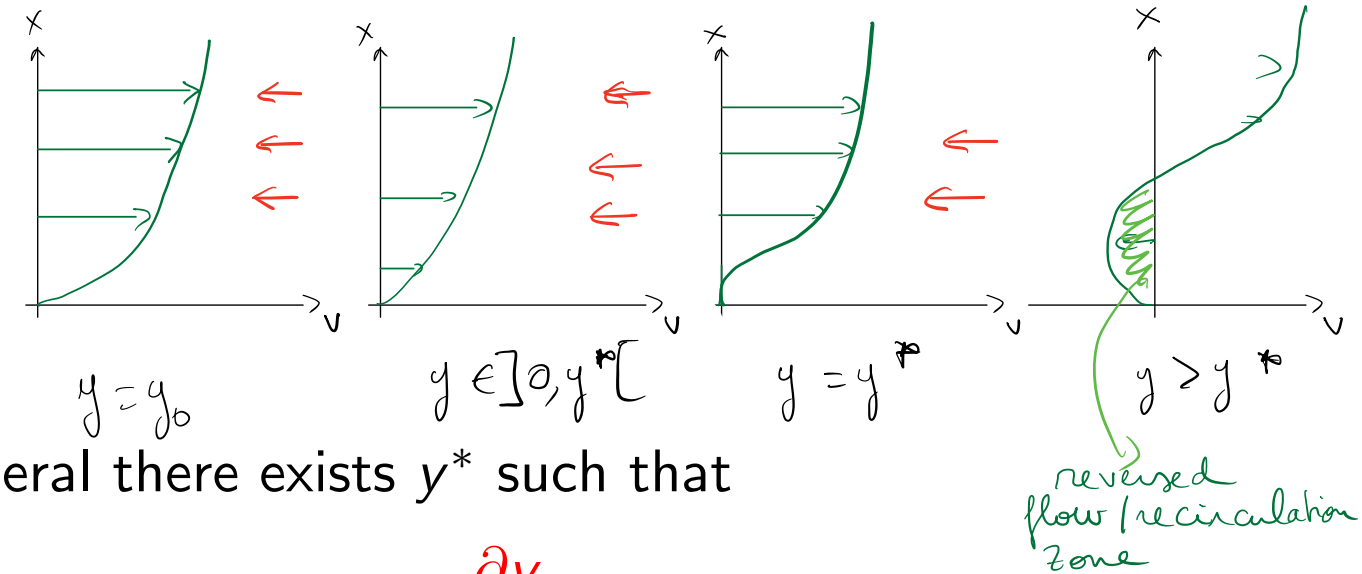
Q&A around the Prandtl-like equation

$$v\partial_y v + u\partial_X v + \Phi - \partial_X^2 v = \overline{\psi}(0, y),$$
$$(u, v) = \nabla^{\perp}_{X,y}\Phi.$$

- Well-posedness when there is no recirculation, i.e. when $v > 0$?
  ✓ [D., Paddick, '21]
- Separation? Goldstein type singularity?
  ✗ Due to loss of scaling invariance.
- Well-posedness in recirculation zone?
  $\sim$ ✓ [D., Marbach, Rax '22; Iyer,Masmoudi, '22]

About BL separation for the Prandtl equation

Setting: monotone solutions of Prandtl with adverse pressure gradient (negative r.h.s)



In general there exists y^* such that

$$\frac{\partial v}{\partial X}_{|y=y^*, X=0} = 0.$$

For $y > y^*$, $X \lesssim 1$, v takes negative values: reversed flow near the boundary.

$\rightarrow$ There exists a curve $X = F(y)$ such that $v(F(y), y) = 0$: separation of the boundary layer.

Definition: y^* is called the separation point.

About separation for (PQG)

Crucial phenomenon for oceanographic applications: eastward jet of the Gulf Stream.
Separation point is often poorly predicted by ocean models.
... But it is unclear that the separation mechanism is the same as Prandtl:

- Lack of scaling invariance;
- Separation & recirculation linearly driven at main order;
- Possible importance of other (3d) effects: topography, startification...

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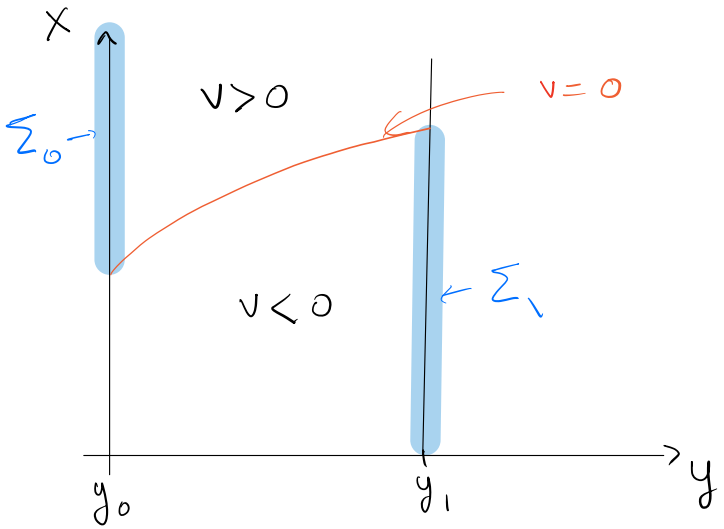
Beyond separation: recirculation

General idea: construct solutions of Prandtl or (PQG) in the vicinity of semi-explicit flows.



Zone $\{v > 0\}$: forward parabolic equation.
Zone $\{v < 0\}$: backward parabolic equation.

$\rightarrow$ Lateral boundary data on blue part of boundary ($\Sigma_0 \cup \Sigma_1$).

⚠The line $\{v = 0\}$ is an unknown of the problem! ($\sim$ free boundary)

Difficulties

Most apparent difficulties:

- **Nonlocality** of the equations: even the linearized equation can be difficult to solve.
- **Nonlinearity and importance of the geometry:** free boundary problem;
- Potential **loss of derivatives** and need for high regularity estimates to construct a strong solution.

$\rightarrow$ Introduction of "good unknowns" in which eq. takes a very simple form (straighten free boundary, vorticoty form of eq.)

Less apparent difficulty:

Need for nonlinear orthogonality conditions ($\Leftrightarrow$ finite codimensional space of data) to avoid singularities near endpoinds of Σ_i.

Indicates that formulation of the problem is probably wrong!

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Motivation: accurate latitude of Gulf Stream separation

- Northward bias in many low resolution GCMs;
- Mechanism remains poorly understood

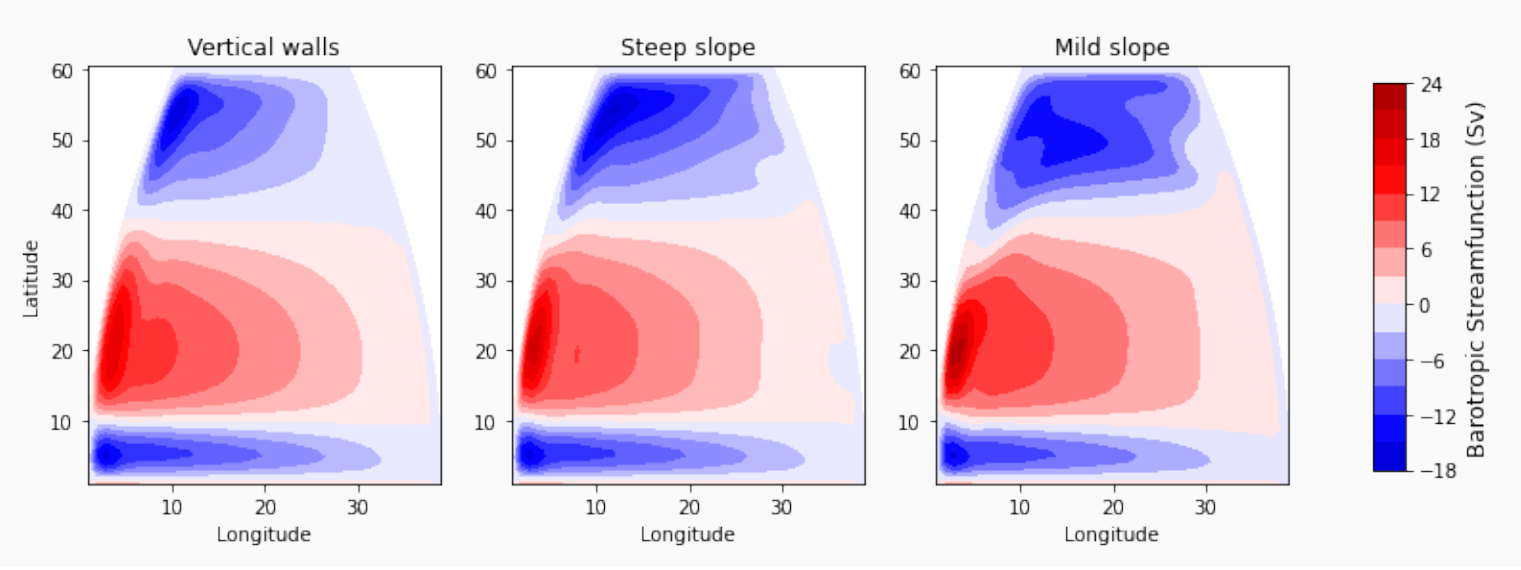
Idea: use mathematical theory to improve description of small scales.

Need to **depart from 2d homogeneous models**:

- Preliminary work by C. Gentil on NEMO showing effect of topography on separation point
- Homogeneous models cannot handle large topographies
- Oceanographic studies on importance of JEBAR term (joint effect of baroclinicity and relief). But no prognostic model.

Preliminary work on NEMO

NEMO simulations by Corentin Gentil:
Idealized "basin" configuration: simple topography with vertical/strong/mild slopes:



→ Confirms the role of topography in the latitude of the Gulf Stream detachment.

Linear 3d model for WBC

Starting point: linearized Navier–Stokes–Boussinesq, strong rotation, strong stratification, above inclined plane.
Asymptotic analysis: pressure

$$p \simeq p^0 + \mathrm{Ro}\, p^1 + \cdots + \mathrm{Ro}^k p^k + O(\mathrm{Ro}^{k+1})$$

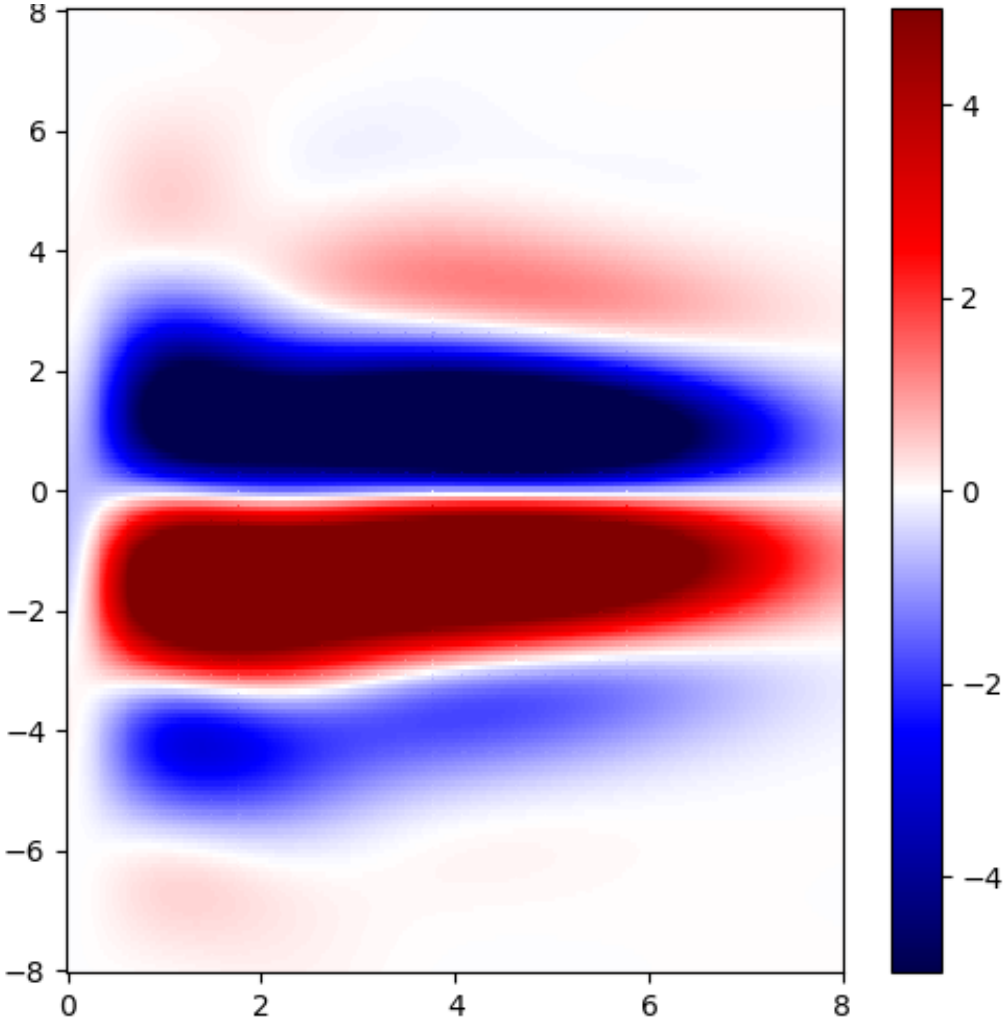
Ro: Rossby number, $\ll 1$.
Result: each term p^j solves

$$\partial_t \Delta p^j + \beta \partial_x p^j - \mathfrak{E} \Delta_h^2 p^j = f^j, \quad \text{\& BC on } p^j, \partial_n p^j$$

where each f^j & BC depend explicitly on lower order terms.
$\rightarrow$ Iterative construction.
Munk type layers: similar expression to 2d case, but with topographic effects. East/West dissymetry.

Plot of pressure at order one $p^0 + \mathrm{Ro}p^1$

(in Python: everything is explicit!)



$\rightarrow$ Separation is visible in first order corrector!

Summary

- Zoology of different behaviors for western boundary layers;
- Excellent playground for mathematicians;
- Nontrivial issues both in linear and nonlinear regimes;
- Need to further explore 3d effects.

THANK YOU FOR YOUR ATTENTION!