On High-Dimensional Mathematics

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- **Segre jets**
- **Hyperbolicity, ampleness**
- **6 classes of CR manifolds of dimension** $\leqslant 5$
- **Explicit Cartan curvatures in CR geometry**
- **CR-Umbilical Locus of Ellipsoids**

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Jets of Segre varieties

- **Real submanifold:**

$$M^{2n+c} \subset \mathbb{C}^{n+c},$$

of codimension:

$$c = \mathsf{codim}\, M.$$

- **Ambient dimension:**

$$\mathrm{N} := n + c.$$

- **CR-genericity condition:**

$$TM + \sqrt{-1}\, TM = T\mathbb{C}^{\mathrm{N}}\big|_M.$$

- **Hence:**

$$T^c M := TM \cap \sqrt{-1}\, TM$$

has constant:

$$\mathsf{rank}\, (T^c M) = n.$$

- **Call:**

$$\mathsf{CRdim}\, M = n.$$

- **Assume real analyticity:** $M \in \mathscr{C}^{\omega}$.
- **Existence of graphing coordinates:**

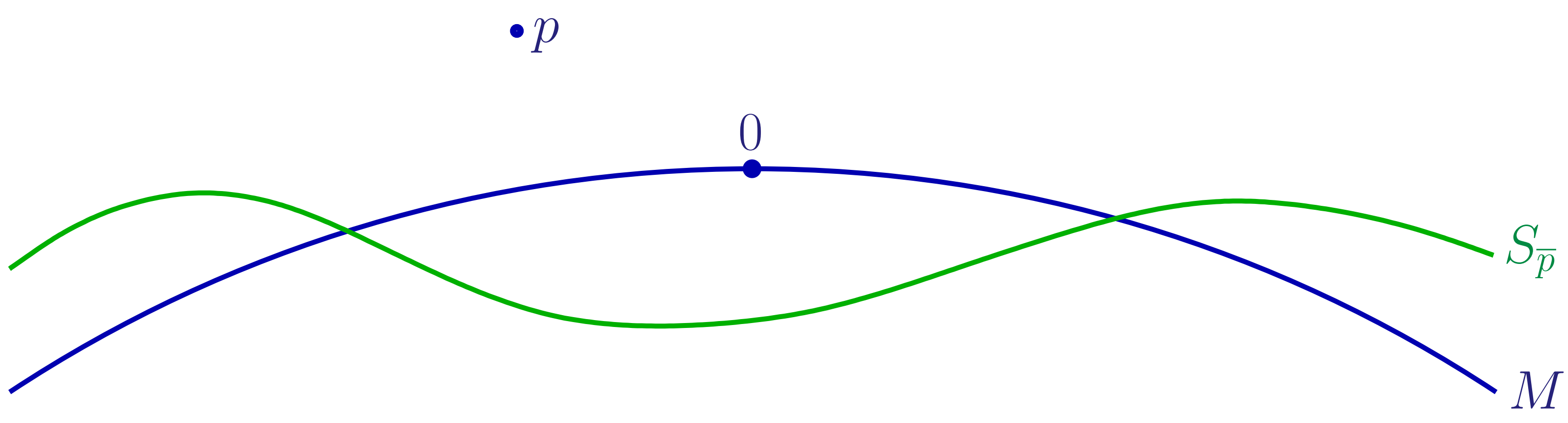
$$\big(z,\,w\big) \;=\; \big(z_1,\ldots,z_n,\;w_1,\ldots,w_c\big),$$

in which M is:

$$w \;=\; \Theta\big(z,\overline{z},\overline{w}\big).$$

- **Equivalently after conjugating:**

$$\overline{w} \;=\; \overline{\Theta}\big(\overline{z},z,w\big).$$



- **May assume:**

$$0 \;\in\; M.$$

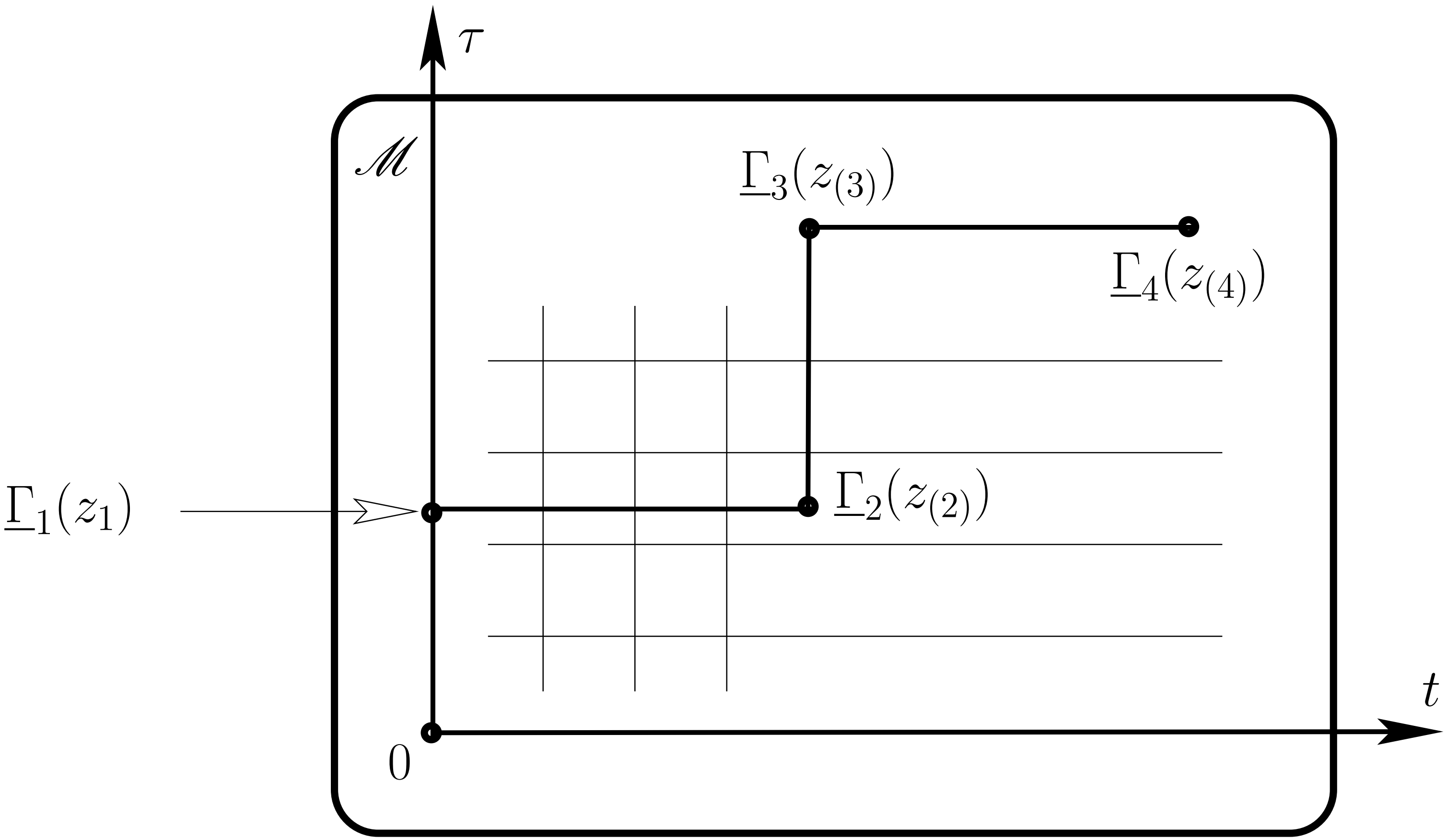
- **Point near the origin:**

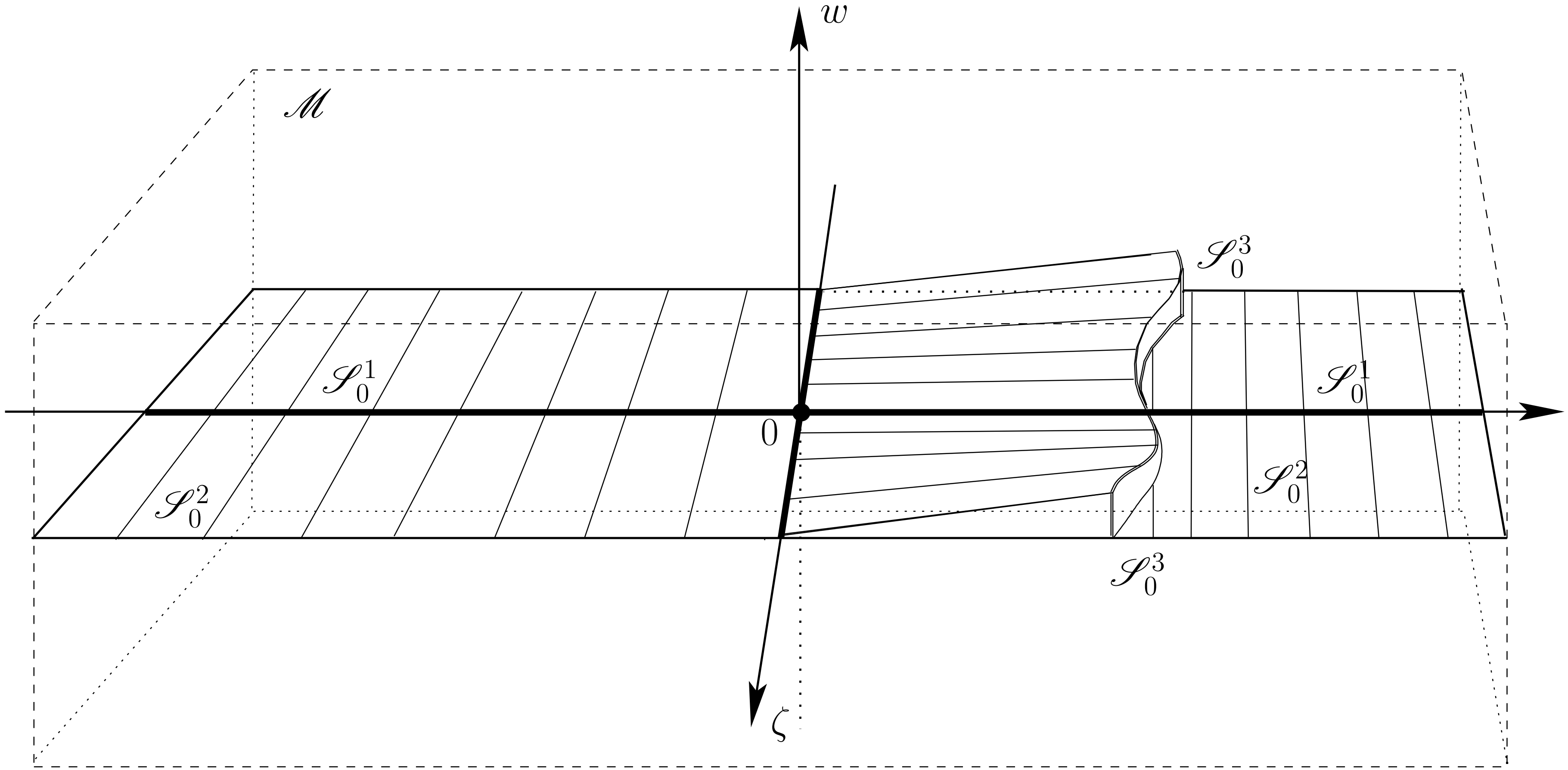
$$p = (z_p, w_p).$$

• **Segre varieties:**

$$S_{\overline{p}} := \Big\{(z,w) \in \mathbb{C}^{\mathrm{N}} \colon \underbrace{w = \Theta\big(z, \overline{z}_p, \overline{w}_p\big)}_{\text{holomorphic!}}\Big\},$$

hence they form a family of complex submanifolds of dimension n.





- **κ-jets:**

$$j^\kappa S_{\overline{p}}.$$

- **κ-jet map:**

$$\begin{aligned}\big(\zeta, z, w\big) &\longmapsto \Big(\zeta,\, \overline{\Theta}(\zeta, z, w), \ldots\ldots, \partial_\zeta^\kappa \overline{\Theta}(\zeta, z, w)\Big)\\ \mathbb{C}^{n+n+c} &\longmapsto \mathbb{C}^{\mathbf{M}}.\end{aligned}$$

Lemma. *A CR-generic manifold is Levi nondegenerate if and only if:*

$$\mathsf{Jet}^1(\mathsf{Segre})$$

is of full rank:

$$n + n + c.$$

- **CR mapping:**

$$h\colon \quad M \longrightarrow M'.$$

- **Another CR-generic:**

$$M' \subset \mathbb{C}^{n+c} \ni (z', w').$$

- **Graphed as:**

$$w' = \Theta'(z', \overline{z}', \overline{w}').$$

- **Conjugate:**

$$\xi' = \overline{\Theta}'(\zeta', z', w').$$

- **CR-map:**

$$(z, w) \longmapsto h(z, w) = \big(f(z, w),\, g(z, w)\big) = (z', w').$$

- **Invariant and unifying concept:**

Definition. [M. 1996] The reflection mapping associated to h is:

$$\xi' - \overline{\Theta}'\big(\zeta', h(z, w)\big).$$

Theorem. [Merker 2000] *Let h be a formal CR map which is:*

- ☐ *either an equivalence;*
- ☐ *or CR-transversal.*

If TM is generated by Lie brackets of $T^c M$, then:

$$\xi' - \Theta'\big(\zeta', h(z, w)\big)$$

is convergent.

Theorem. [Merker 2000] *If M' is holomorphically nondegenerate, h itself is convergent.*

• **Observation:** This was the final result when TM is generated by Lie brackets of $T^c M$, because of the naturality of a necessary condition. $^\infty$

□ M holomorphically *degenerate* means the existence of:

$$L = \text{holomorphic vector field tangent to } M.$$

□ Then the flow:

$$\mathsf{exp}\big(\widehat{a} \cdot L\big)$$

where $\widehat{a}$ is any *non-convergent* formal series is a *non-convergent* invertible formal CR equivalence $M \longrightarrow M$. □

• **Observation:** The κ-jet map $\mathsf{Jet}^\kappa\big(\mathsf{Segre}\big)$ is **not** assumed to be of constant rank.

Artin

Hironaka

• **Observation:** The reflection mapping:

$$\xi' - \overline{\Theta}'\big(\zeta', h(z, w)\big)$$

is an invariant object *which expresses explicitly in terms of a defining functions for the CR manifold* $M^{2n+c} \subset \mathbb{C}^{n+c}$.

• **Universality:** Expressing geometric objects in terms of basic data is universal.

Hyperbolicity, ampleness

- **Consider a generic complex algebraic hypersurface:**

$$\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C}).$$

- **Dimension:**

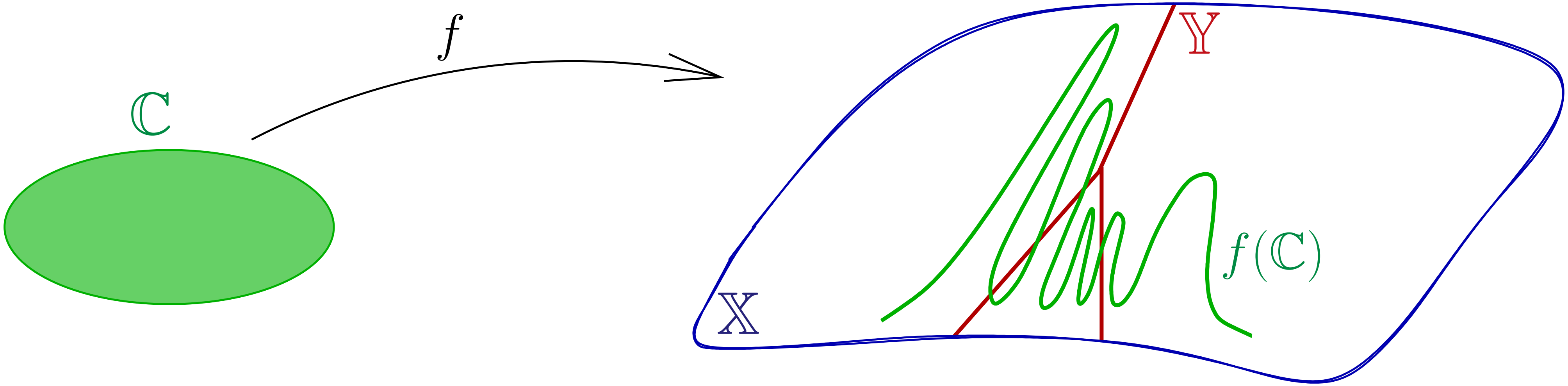
$$n := \mathsf{dim}\, \mathbb{X}.$$

- **Degree:**

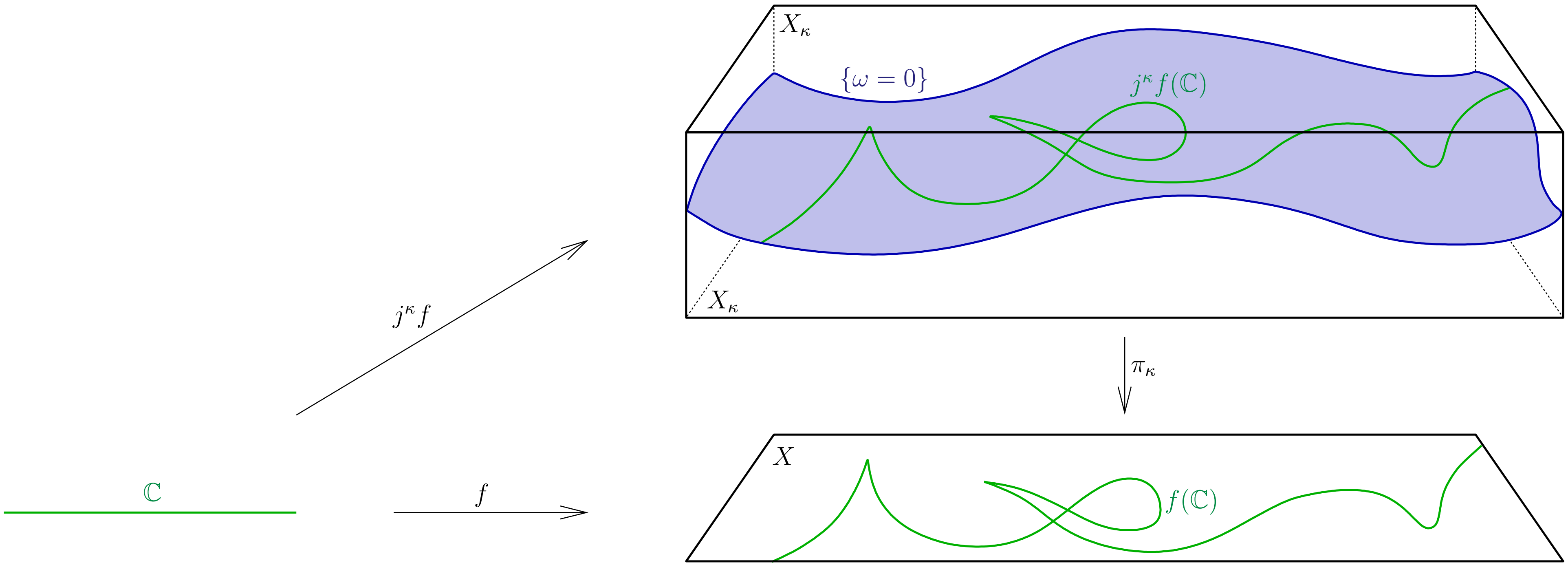
$$d := \mathsf{deg}\, \mathbb{X}.$$

- **Look at:** Nonconstant entire holomorphic curves:

$$f\colon \quad \mathbb{C} \longrightarrow \mathbb{X}.$$



- **Method:** Lift f to the κ-jet space:



Theorem. [Diverio-Merker-Rousseau, Invent. 2010] *For generic* $\mathbb{X}$*, there exists a proper algebraic subvariety:*

$$\mathbb{Y} \subsetneqq \mathbb{X}$$

inside which all nonconstant entire holomorphic curves:

$$f\colon \quad \mathbb{C} \longrightarrow \mathbb{X}$$

are in fact contained:

$$f(\mathbb{C}) \subset \mathbb{Y},$$

provided that:

$$\deg \mathbb{X} = d \geqslant \mathbf{2^{n^5}}.$$

- **Improvement:**

Theorem. [Darondeau, Ph.D. Orsay, IMRN 2016] *Same conclusion for:*

$$d \geqslant (\mathbf{5n})^{\mathbf{2}}\,\mathbf{n^n}.$$

$$Merker_2 := 4294967296$$

$Merker_3 :=$

1413477651822707463666380005943348126619871175004951664972849610340958208

$Merker_4 :=$

179769313486231590772930519078902473361797697894230657273430081157732675805\
500963132708477322407536021120113879871393357658789768814416622492847430639\
474124377767893424865485276302219601246094119453082952085005768838150682\
342462881473913110540827237163350510684586298239947245938479716304835356329\
624224137216

$Merker_5 :=$

523282787918350852835205722897064609275976654481364715188209859886168997863\
538787568456115520655575006398497633467321503793224008528863465412684209666\
278555177105547830411415410831286843828684716586961010653695549005350865874\
417589695912820305460533332381770025796116190261203851369723845099986716066\
552514989559461379435496613712352565322877746694768802791398512815343301066\
631642527809678764553511630464647308610733651478034430451188431614291992522\
117746296212627046461415455761835855306788293241198204327000036046731852072\
241264772017502295762617185634849421228112540013406827869793087656368075872\
854773375492957484072736071541304891129456351912648705044047798247177147474\

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> for n from 2 to 5 do Darondeau[n] := (5*n)^2*n^n end do;
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$$Darondeau_2 := 400$$

$$Darondeau_3 := 6075$$

$$Darondeau_4 := 102400$$

$$Darondeau_5 := 1953125$$

- **Open Problem:**

$$d \overset{?}{\geqslant} \mathbf{constant}^{\mathbf{n}}.$$

• **Study:** Product of multidimensional Laurent series:

$$\mathbf{A} \cdot \mathbf{E} \cdot \mathbf{F},$$

where:

$$\mathbf{A} := \sum_{\substack{0 \leqslant k_2 \leqslant n \\ 0 \leqslant k_3 \leqslant n+k_2 \\ \cdots\cdots\cdots\cdots \\ 0 \leqslant k_n \leqslant n+k_{n-1}}} \frac{(n^2)!}{(n+k_n)!\,(n+k_{n-1}-k_n)!\,\cdots\,(n+k_2-k_3)!\,(n-k_2)!}$$

$$\mathbf{r}^{\frac{n(n-1)}{2}-k_2-\cdots-k_n}\,\frac{1}{(w_2)^{k_2}\cdots(w_n)^{k_n}},$$

with $\mathbf{r} \geqslant 3$, where:

$$\mathbf{E} = \frac{1-w_2}{1-2w_2}\,\frac{1-w_2w_3}{1-2w_2w_3}\,\cdots\cdots\cdots\,\frac{1-w_2w_3w_4\cdots w_n}{1-2\,w_2w_3w_4\cdots w_n},$$

and where:

$$\mathbf{F} := \frac{1-w_3}{1-2w_3+w_2w_3}\,\frac{1-w_3w_4}{1-2w_3w_4+w_2w_3w_4}\,\cdots\cdots\,\frac{1-w_3\cdots w_n}{1-2w_3w_4\cdots w_n+w_2\cdots w_n}$$

$$\frac{1-w_4}{1-2w_4+w_3w_4}\,\cdots\cdots\cdots\,\frac{1-w_4\cdots w_n}{1-2w_4\cdots w_n+w_3w_4\cdots w_n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,\frac{1-w_n}{1-2w_n+w_{n-1}w_n}.$$

• **Computational aspects:** Simultaneous presence of positive contributions and of negative contributions.

$$
\begin{aligned}
FFFF_5 := {} & 4\, w_2^2 w_3^3 + w_2^2 w_3^2 w_4 - 8\, w_2 w_3^4 + 2\, w_2 w_3^3 w_4 + w_2 w_3^2 w_4^2 + 3\, w_2 w_3^2 w_4 w_5 + 2\, w_2 \\
& w_3^2 w_5 w_6 + 2\, w_2 w_3^2 w_6 w_7 + 2\, w_2 w_3^2 w_7 w_8 + w_2 w_3 w_4^2 w_5 + 2\, w_2 w_3 w_4 w_5^2 + w_2 w_3 w_4 w_5 w_6 \\
& + w_2 w_3 w_4 w_6 w_7 + w_2 w_3 w_4 w_7 w_8 + 2\, w_2 w_3 w_5 w_6^2 + w_2 w_3 w_5 w_6 w_7 + 2\, w_2 w_3 w_6 w_7^2 \\
& + w_2 w_3 w_6 w_7 w_8 + 2\, w_2 w_3 w_7 w_8^2 + 4\, w_3^2 w_4^3 + w_3^2 w_4^2 w_5 - 8\, w_3 w_4^4 + 2\, w_3 w_4^3 w_5 + w_3 w_4^2 \\
& w_5^2 + 3\, w_3 w_4^2 w_5 w_6 + 2\, w_3 w_4^2 w_6 w_7 + 2\, w_3 w_4^2 w_7 w_8 + w_3 w_4 w_5^2 w_6 + 2\, w_3 w_4 w_5 w_6^2 \\
& + w_3 w_4 w_5 w_6 w_7 + w_3 w_4 w_5 w_7 w_8 + 2\, w_3 w_4 w_6 w_7^2 + w_3 w_4 w_6 w_7 w_8 + 2\, w_3 w_4 w_7 w_8^2 \\
& + 4\, w_4^2 w_5^3 + w_4^2 w_5^2 w_6 - 8\, w_4 w_5^4 + 2\, w_4 w_5^3 w_6 + w_4 w_5^2 w_6^2 + 3\, w_4 w_5^2 w_6 w_7 + 2\, w_4 w_5^2 w_7 w_8 \\
& + w_4 w_5 w_6^2 w_7 + 2\, w_4 w_5 w_6 w_7^2 + w_4 w_5 w_6 w_7 w_8 + 2\, w_4 w_5 w_7 w_8^2 + 4\, w_5^2 w_6^3 + w_5^2 w_6^2 w_7 \\
& - 8\, w_5 w_6^4 + 2\, w_5 w_6^3 w_7 + w_5 w_6^2 w_7^2 + 3\, w_5 w_6^2 w_7 w_8 + w_5 w_6 w_7^2 w_8 + 2\, w_5 w_6 w_7 w_8^2 + 4 \\
& w_6^2 w_7^3 + w_6^2 w_7^2 w_8 - 8\, w_6 w_7^4 + 2\, w_6 w_7^3 w_8 + w_6 w_7^2 w_8^2 + 4\, w_7^2 w_8^3 - 8\, w_7 w_8^4
\end{aligned}
$$

• **Observation:** Entire curves $f \colon \mathbb{C} \longrightarrow \mathbb{X}$ satisfy algebraic differential equations:

$$Q\big(f, f', f'', \dots, f^{(\kappa)}\big) \,=\, 0,$$

but nobody on Earth is able to express $Q = Q_P$ in terms of a homogeneous polynomial defining equation:

$$\mathbb{X} \,=\, \Big\{[X_0 \colon X_1 \colon \cdots \colon X_{n+1}] \in \mathbb{P}^{n+1}(\mathbb{C}) \colon\; P\big([X_0 \colon X_1 \colon \cdots \colon X_{n+1}]\big) = 0\Big\}.$$

Low Degree Hyperbolic Hypersurfaces $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$

- **Complex line:**

$$\mathbb{C} \ni z = x + \sqrt{-1}\, y.$$

- **Unit disc:**

$$\mathbb{D} := \big\{ z \in \mathbb{C} \colon |z| < 1 \big\}.$$

- **Euclidean metric:**

$$\big|dz\big|_{\mathsf{Eucl}}^2 := dx^2 + dy^2.$$

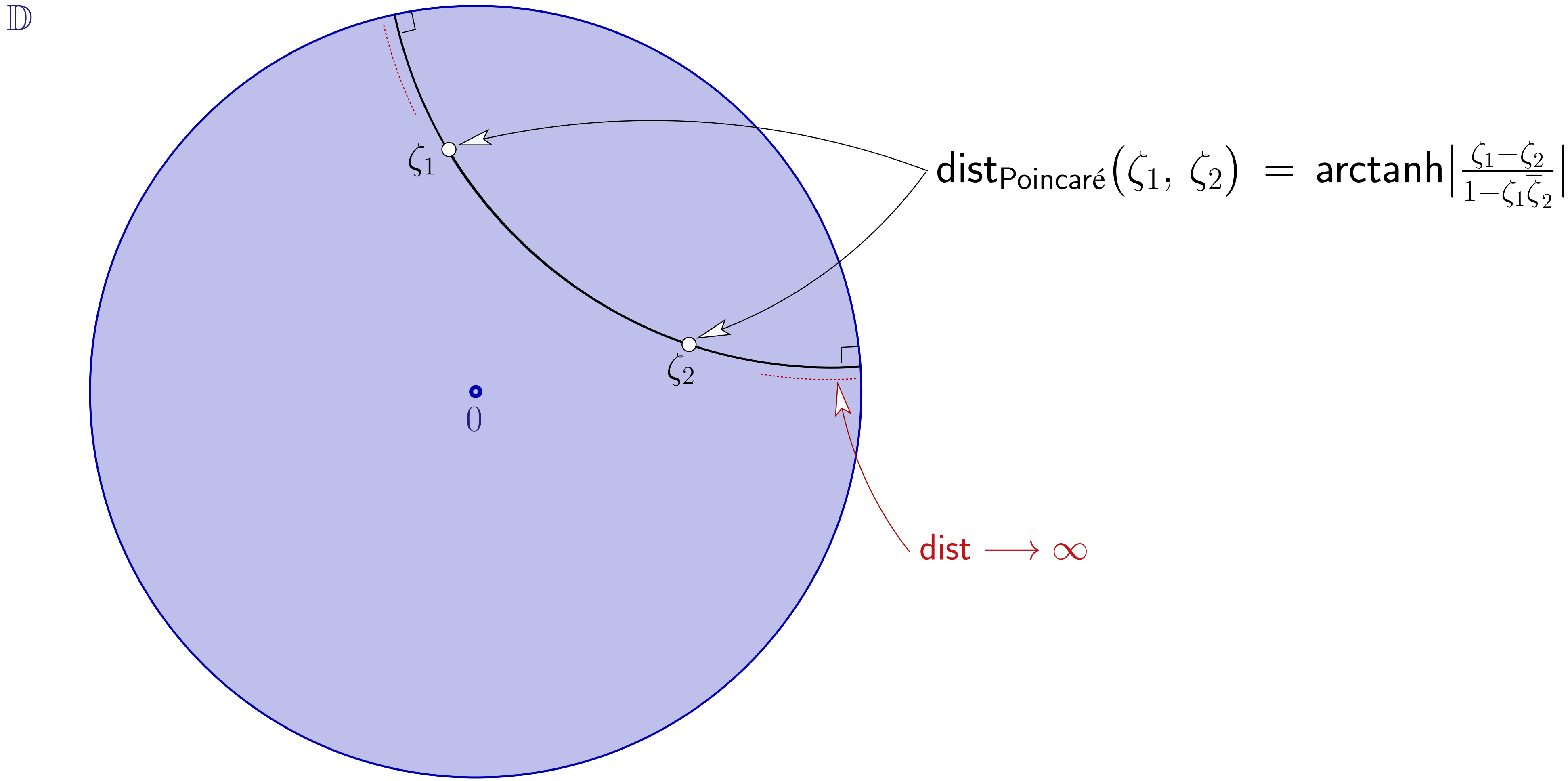
- **Poincaré metric:**

$$\big|dz\big|_{\mathsf{Pcé}}^2 := \frac{\big|dz\big|_{\mathsf{Eucl}}^2}{(1 - |z|^2)^2}.$$

- **Curvature:**

$$\mathsf{curv}_{\mathsf{Pcé}} \equiv -4.$$

- **Geodesics of the Poincaré metric:**



- **Connected complex manifold $\mathbb{X}$:**

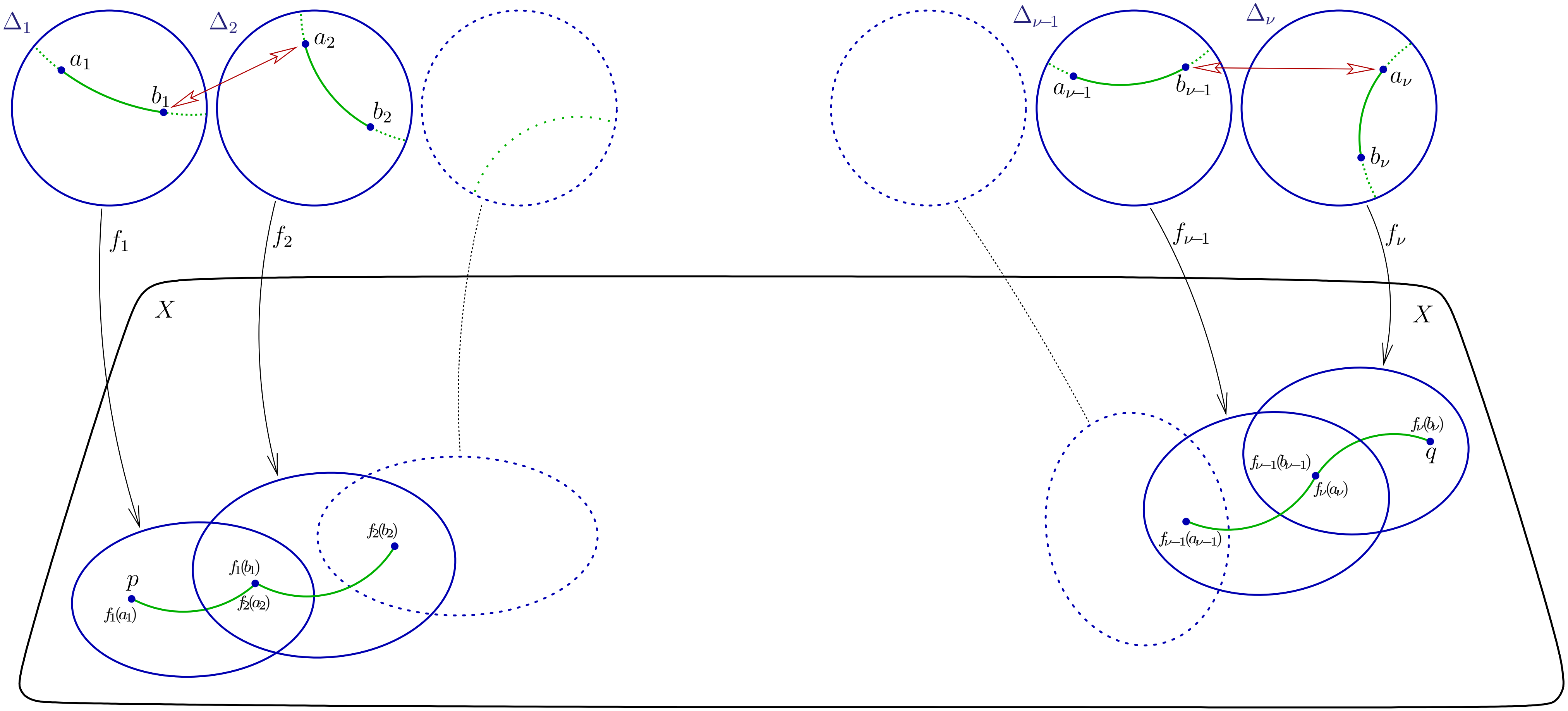
$$\dim \mathbb{X} \ = \ n \ \geqslant \ 1.$$

- **Any two points:**

$$p,\ q \ \in \ \mathbb{X}.$$

• Chain of holomorphic discs:

$$p = f_1(a_1), \quad f_1(b_1) = f_2(a_2), \quad \dots\dots, \quad f_{\nu-1}(b_{\nu-1}) = f_\nu(a_\nu), \quad f_\nu(b_\nu) = q.$$



$$\text{pseudo-dist}_{\,\text{Kobayashi}}(p,q) \;:=\; \inf_{\nu,\, f_1,\dots,f_\nu} \left\{ \sum_{i=1}^{\nu} \text{dist}_{\,\text{Poincaré}}(a_i, b_i) \right\}.$$

• **Definition:** A complex manifold $\mathbb{X}$ is said to be *Kobayashi-hyperbolic* when:

$$(p \neq q) \quad \Longrightarrow \quad \Big(\text{pseudo-dist}_{\,\text{Kobayashi}}(p, q) \, > \, 0\Big).$$

• **Second Theorem of Brody.** For a *compact* complex manifold $\mathbb{X}$:

$$\Big(\mathbb{X} \text{ is Kobayashi-hyperbolic}\Big) \iff \Big(\text{all holomorphic } f \colon \mathbb{C} \longrightarrow X \text{ are constant}\Big).$$

• **Corollary.** Kobayashi-hyperbolicity is stable under small perturbations.

Conjecture. [Kobayashi 1970] *Generic hypersurfaces* $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *with:*

$$\deg \mathbb{X} \, \geqslant \, 2\,n + 1$$

should be hyperbolic.

Theorem. [Brotbek, Invent. Math. 2018] *Generic hypersurfaces* $\mathbb{X}^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ *with:*

$$\deg \mathbb{X} \, \gg \, 1$$

are hyperbolic.

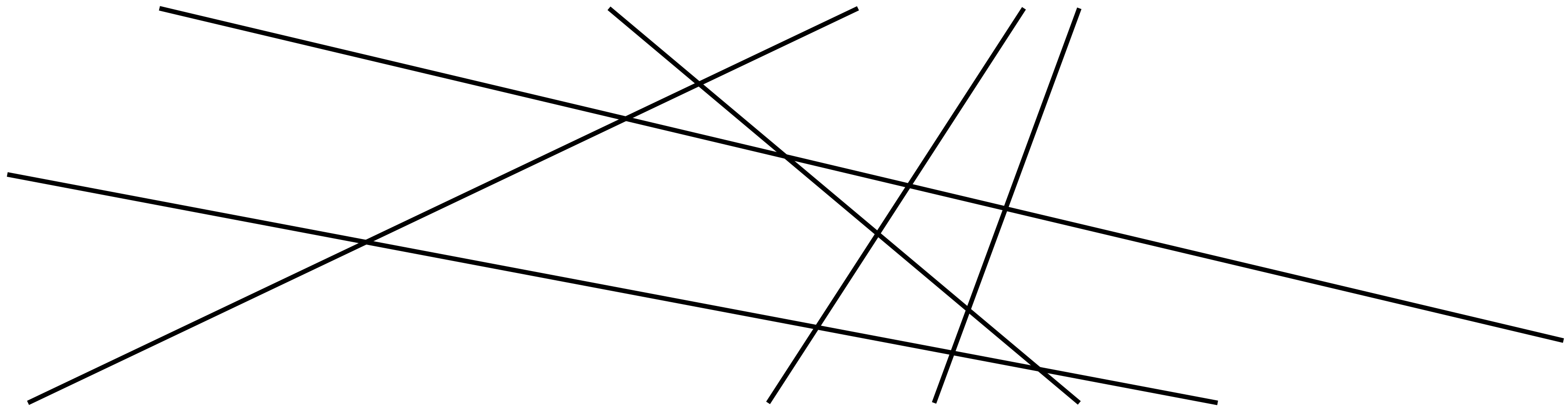
• **Homogeneous coordinates on $\mathbb{P}^{n+1}(\mathbb{C})$:**

$$Z = \big[Z_0 \colon Z_1 \colon \cdots \colon Z_{n+1}\big].$$

• **Collection of $\mathrm{Q} \geqslant 1$ hyperplanes:**

$$H_i := \big\{Z \in \mathbb{P}^{n+1} \colon\ h_i(Z) = 0\big\} \qquad\qquad (1 \leqslant i \leqslant \mathrm{Q}),$$

with $\deg h_i = 1$.



• **General position:**

$$\forall\ I \subset \big\{1, \dots, \mathrm{Q}\big\} \quad \text{with } \operatorname{Card} I = n+2, \qquad \bigcap_{i \in I} H_i = \emptyset.$$

Theorem. [Tuan Huynh, IMRN 2015] *There exists a homogeneous polynomial:*

$$s = s(Z), \qquad \text{deg}\, s = 2n + 2,$$

such that the hypersurface:

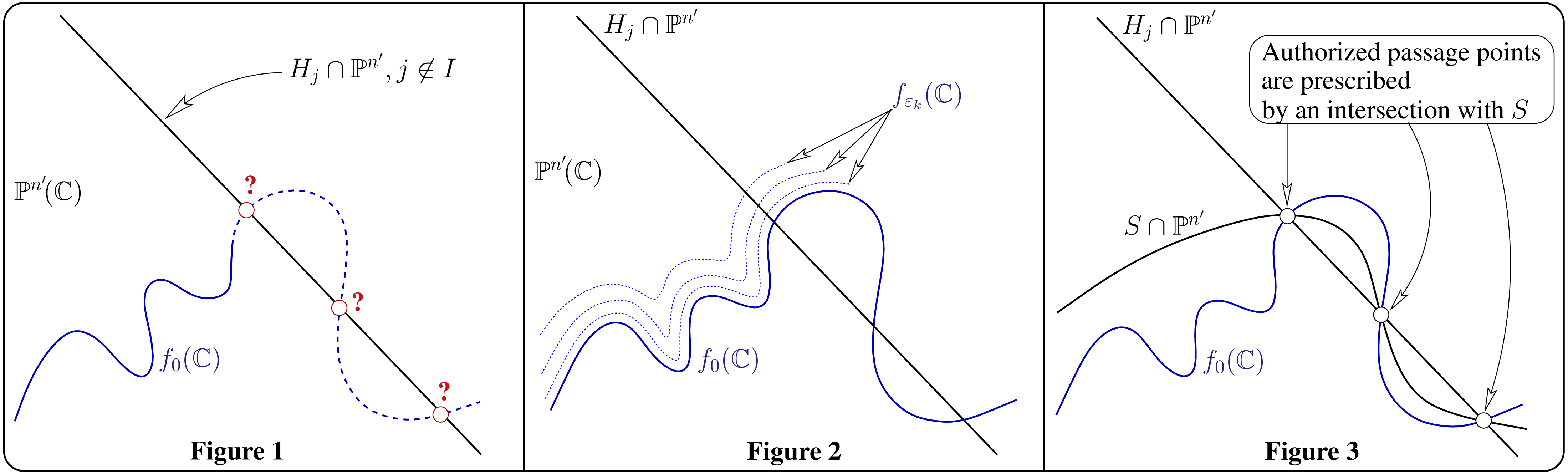
$$h_1(Z) \cdots h_{2n+2}(Z) - \delta\, s(Z) = 0$$

is Kobayashi-hyperbolic for all small nonzero $\delta \in \mathbb{C}$ *and for* $\mathbf{n = 2, 3, 4, 5}$.

- **Open Problem:** How to reach examples in degree:

$$d \geqslant \text{constant} \cdot n\, \mathbf{?}$$

- **The percolation method of Zaidenberg:**



- **Equation in the case $n = 2$:** [Duval]

$$
\begin{aligned}
0 = h_1h_2h_3h_4h_5h_6 - \delta\big(h_1^2h_2^2h_3h_4 &- \varepsilon_{56}\big(h_1^2h_2^2h_3h_4 - \\
&- \varepsilon_{46}\big(h_1^2h_2^2h_3h_6 - \varepsilon_{45}\big(h_1^2h_2^2h_4h_5 - \\
&- \varepsilon_{36}\big(\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots - \\
&- \varepsilon_{26}\big(\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots - \\
&- \varepsilon_{16}\big(\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots - \varepsilon_{13}\big(h_3^2h_4^2h_5h_6 - \varepsilon_{12}s_0\big)\big)\big)\big)
\end{aligned}
$$

- **With 16 extremely small quantities:**

$$
\begin{aligned}
0 < \delta \ll \varepsilon_{56} &\ll \\
&\ll \varepsilon_{46} \ll \varepsilon_{45} \ll \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\ll \varepsilon_{16} \ll \varepsilon_{15} \ll \cdots\cdots\cdots\cdots\cdots \ll \varepsilon_{13} \ll \varepsilon_{12} \ll 1.
\end{aligned}
$$

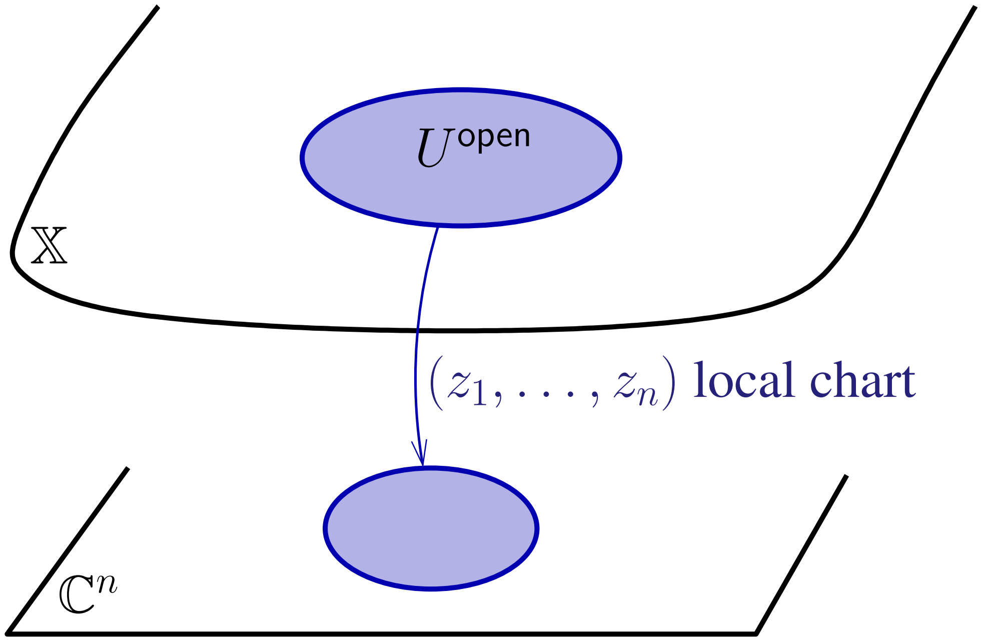
Ampleness of Cotangent Bundles

• **Complex manifold:**

$$\mathbb{X}.$$

• **Dimension:**

$$\dim \mathbb{X} = n \geqslant 1.$$



• **Holomorphic cotangent bundle:**

$$\Omega_{\mathbb{X}}.$$

- **Local section:**

$$\omega \in \Gamma\big(U, \Omega_{\mathbb{X}}\big).$$

- **Writes:**

$$\omega = \omega_1(z)\, dz_1 + \cdots + \omega_n(z)\, dz_n.$$

- **Riemann surfaces:** $[n = 1]$

$$\mathsf{genus} = \mathsf{dim}\, \Gamma\big(\mathbb{X},\, \Omega_{\mathbb{X}}\big) \underbrace{\geqslant 1.}_{\text{often}}$$

- **However, in dimension $n \geqslant 2$:**

$$\underbrace{0 \;=}_{\substack{\text{Naruki} \\ 1977}} \mathsf{dim}\, \Gamma\big(\mathbb{X},\, \Omega_{\mathbb{X}}\big).$$

- **Substitute:**

$$\Omega_{\mathbb{X}} \dashrightarrow \mathsf{Sym}^k \Omega_{\mathbb{X}}.$$

- **Bundle of symmetric k-differentials:** With $k \geqslant 1$:

• **Local section:**

$$\omega = \sum_{1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n} \omega_{i_1,\dots,i_k}(z)\, \underbrace{dz_{i_1}\odot\cdots\odot dz_{i_k}}_{\substack{\text{symmetric tensor } \otimes \\ \mathsf{a}\odot\mathsf{b}=\mathsf{b}\odot\mathsf{a}}}.$$

• **Always in what follows:**

$$\mathbb{X} = \mathbb{X}^n \subset \mathbb{P}^{n+c}(\mathbb{C}).$$

• **Smooth complete intersection of projective algebraic hypersurfaces:**

$$\mathbb{X} = \mathbb{H}_1 \cap \cdots \cap \mathbb{H}_c,$$

with:

$$c = \operatorname{codim} \mathbb{X}.$$

• **Hypersurfaces:**

$$\mathbb{H}_i = \Big\{[Z] \in \mathbb{P}^{n+c}\colon\ R_i\big(Z_0\colon Z_1\colon\ \cdots\colon Z_{n+c}\big) = 0\Big\}.$$

• **Abbreviate:**

$$\mathrm{N} = n + c.$$

• Homogeneous polynomials:

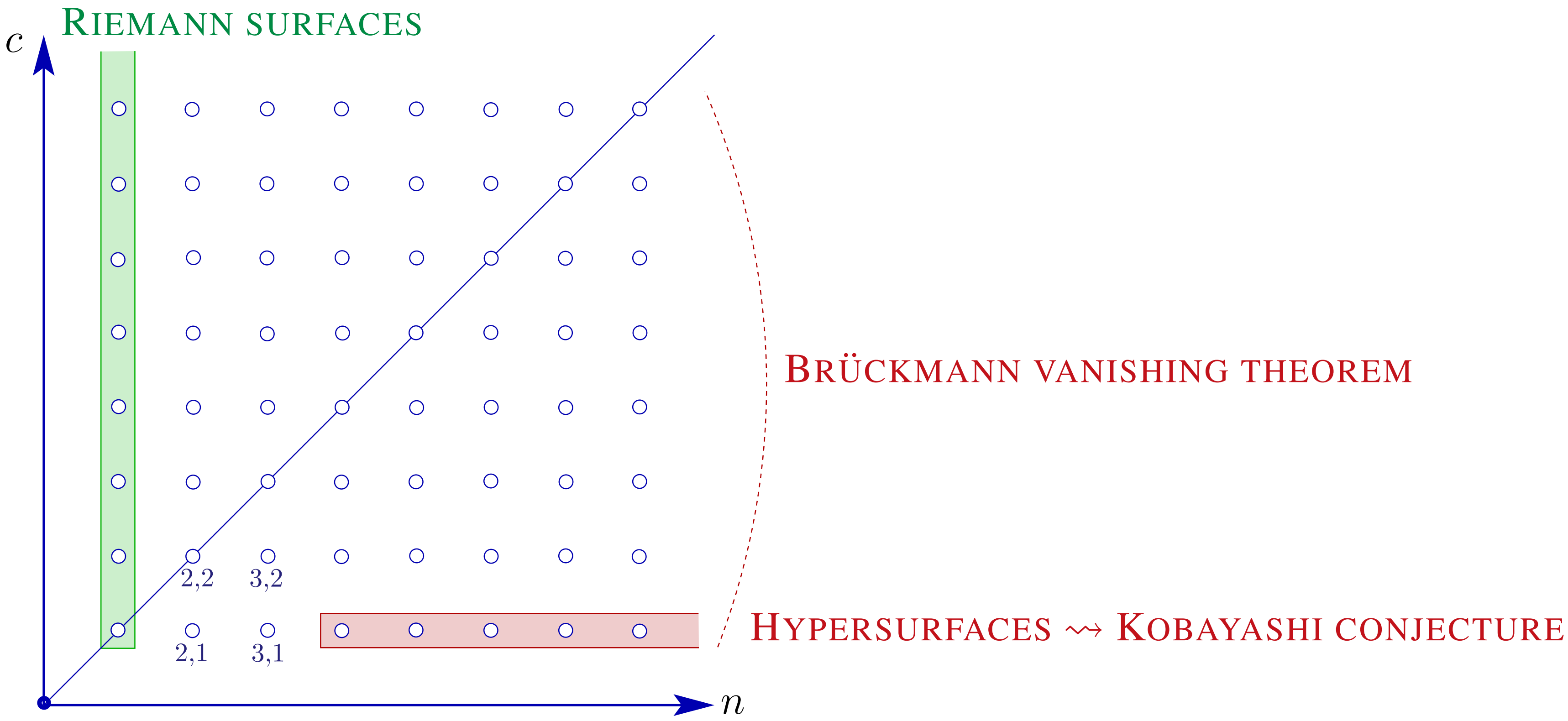
$$R_i = \sum_{a_0+a_1+\cdots+a_N=d_i} \underbrace{R_{i,a_0,a_1,\dots,a_N}}_{\in\,\mathbb{C}} \left(Z_0\right)^{a_0}\left(Z_1\right)^{a_1}\cdots\cdots\left(Z_N\right)^{a_N}.$$

As said, by Naruki, there are often no nonzero global holomorphic sections of $\Omega_{\mathbb{X}}$.

Even worse:

Brückmann Vanishing Theorem. [1996] *When $c < n$, for all $k \geqslant 1$:*

$$0 = \Gamma\Big(\mathbb{X}, \mathsf{Sym}^k\Omega_{\mathbb{X}}\Big).$$



Fortunately:

Theorem. [Schneider, Brotbek] *When the codimension $c \geqslant n$ is larger than the dimension:*

$$\dim \Gamma\Big(\mathbb{X},\, \mathrm{Sym}^k \Omega_{\mathbb{X}}\Big) \;\geqslant\; \underset{\text{positive}}{\text{constant}} \cdot k^{2n-1},$$

for all $k \geqslant {}^{\exists}k_0 \gg 1$.

• **Interpretation:** This means *bigness* of the line bundle:

$$\begin{array}{c} \mathscr{O}_{\mathbb{P}(\Omega_{\mathbb{X}})}(1) \\ \big\downarrow \\ \mathbb{P}(\Omega_{\mathbb{X}}). \end{array}$$

General idea: Bigness conducts to expect maximal richness of the space of sections

Conjecture. [Debarre 2005, ENS Paris] *When $c \geqslant n$, the intersection of generic hypersurfaces:*

$$\mathbb{H}_1 \cap \cdots \cap \mathbb{H}_c =: \mathbb{X} \subset \mathbb{P}^{n+c},$$

of sufficiently high degrees:

$$d_1, \ldots\ldots, d_c \gg 1$$

should have cotangent bundle $\Omega_{\mathbb{X}}$ ample.

• **Definition:** [Hartshorne 1966] $\Omega_{\mathbb{X}}$ is called *ample* when there exists $k_0 \gg 1$ such that, for all $k \geqslant k_0$, the bundle:

$$\mathsf{Sym}^k \Omega_{\mathbb{X}} \longrightarrow \mathbb{X}$$

is *very ample*.

• **Definition:** $\mathsf{Sym}^k \Omega_{\mathbb{X}}$ is called *very ample* when two conditions hold.

(i) For all $x_1 \neq x_2 \in \mathbb{X}$, global sections are surjective on fibers:

$$\Gamma\Big(\mathbb{X},\, \mathsf{Sym}^k \Omega_{\mathbb{X}}\Big) \xrightarrow[\text{onto}]{} \mathsf{Sym}^k \Omega_{\mathbb{X}}\Big|_{x_1} \oplus \mathsf{Sym}^k \Omega_{\mathbb{X}}\Big|_{x_2}.$$

(ii) For all $x \in \mathbb{X}$, global sections are surjective on first jets:

$$\Gamma\Big(\mathbb{X},\, \mathsf{Sym}^k \Omega_{\mathbb{X}}\Big) \xrightarrow[\text{onto}]{} \mathsf{Jet}_1 \mathsf{Sym}^k \Omega_{\mathbb{X}}\Big|_{x}.$$

Theorem. [XIX$^{\text{th}}$ Century] *Riemann surfaces* $(n = 1)$ *of genus* $\geqslant 2$ *have cotangent bundle* $\Omega_{\mathbb{X}}$ *very ample.*

[Complete solution of Debarre Ampleness Conjecture]

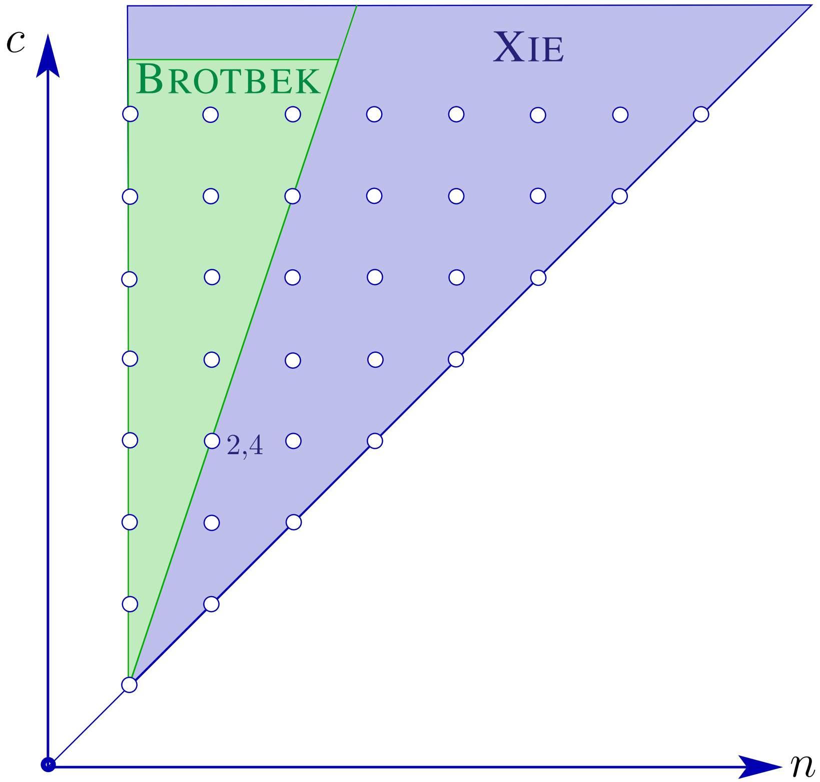
Theorem. [Xie 2015, Ph.D. Orsay, Invent. Math. 2018] *For all* $c \geqslant n \geqslant 1$, *for generic:*

$$\mathbb{H}_1 \cap \cdots \cap \mathbb{H}_c = \mathbb{X} \subset \mathbb{P}^{n+c=\mathrm{N}},$$

of degrees: [Effective!]

$$d_1, \ldots\ldots, d_c \geqslant \mathrm{N}^{\mathrm{N}^2},$$

$\Omega_{\mathbb{X}}$ *is ample.*



- **Before:**

Theorem. [Brotbek, Math. Ann. 2014] *For $c \geqslant 3n-2$ and for equal degrees:*

$$d_1 = \cdots\cdots = d_c \geqslant 2(n+c)+3,$$

$\Omega_{\mathbb{X}}$ is ample.

Brotbek-Darondeau 2015: Noneffective solution

Theorem. [M. 2015, using multidimensional resultants] *Improvement on Xie's degree bound:*

$$d_1, \ldots\ldots, d_c \geqslant \mathrm{N}^{5\,\mathrm{N}}.$$

Theorem. [M. 2016] *The surface* $\mathbb{X}^2 \subset \mathbb{P}^6(\mathbb{C})$ *having equations:*

$$Z_1^{15} = \big(3T + 4X + 6Y\big)\, T^{15} + \big(2T + 5X + Y\big)\, X^{15} + \big(7T + 2X + 8Y\big)\, Y^{15}$$
$$Z_2^{15} = \big(5T + 2X + 9Y\big)\, T^{15} + \big(T + X + Y\big)\, X^{15} + \big(3T + 6X + 3Y\big)\, Y^{15}$$
$$Z_3^{15} = \big(2T + 7X + 2Y\big)\, T^{15} + \big(T + 6X + Y\big)\, X^{15} + \big(8T + X + 2Y\big)\, Y^{15}$$
$$Z_4^{15} = \big(6T + 3X + 9Y\big)\, T^{15} + \big(2T + 2X + Y\big)\, X^{15} + \big(4T + X + 4Y\big)\, Y^{15}$$

has ample cotangent bundle outside the hyperplanes $\{Z_i = 0\}$.

- **Ideas of proof:** Take a large integer:

$$e \gg 1.$$

- **Fermat-type complete intersections:**

$$X = \{F_1 = 0\} \cap \cdots \cap \{F_c = 0\},$$

where:

$$F_j(Z) = A_{j,0}(Z) \cdot (Z_0)^e + \cdots + A_{j,N}(Z) \cdot (Z_N)^e,$$

where:

$$A_{j,i}(Z) \in \mathbb{C}_{\mathsf{hom}}[Z]$$

say:

$$\mathsf{deg}\, A_{j,i}(Z) = 1.$$

- **All degrees are equal:**

$$d_1 = \cdots = d_c = 1 + e =: d.$$

- **Consider one equation (drop indices):**

$$0 = A_0\, Z_0^e + A_1\, Z_1^e + \cdots + A_N\, Z_N^e.$$

- **Employ jet notation:**

$$dZ_i \;\longleftrightarrow\; Z_i'.$$

- **Differentiate once:**

$$0 \;=\; \big(Z_0 A_0' + e Z_0' A_0\big) Z_0^{e-1} + \cdots + \big(Z_N A_N' + e Z_N' A_N\big) Z_N^{e-1}.$$

- **View equations by pairs:** $1 \leqslant j \leqslant c$:

$$\begin{aligned} 0 &= \big(A_{j,0}\, Z_0\big)\, Z_0^{e-1} + \cdots\cdots\cdots\cdots + \big(A_{j,N}\, Z_N\big)\, Z_N^{e-1}, \\ 0 &= \big(Z_0 A_{j,0}' + e Z_0' A_{j,0}\big) Z_0^{e-1} + \cdots + \big(Z_N A_{j,N}' + e Z_N' A_{j,N}\big) Z_N^{e-1}. \end{aligned}$$

- **Treat the enlightening case of** $X^1 \subset \mathbb{P}^2$**:**

$$\begin{aligned} 0 &= \big(Z_0 A_0\big)\, Z_0^{e-1} + \big(Z_1 A_1\big)\, Z_1^{e-1} + \big(Z_2 A_2\big)\, Z_2^{e-1}, \\ 0 &= \big(Z_0 A_0' + e Z_0' A_0\big) Z_0^{e-1} + \big(Z_1 A_1' + e Z_1' A_1\big) Z_1^{e-1} + \big(Z_2 A_2' + e Z_2' A_2\big) Z_2^{e-1}. \end{aligned}$$

- **Abbreviate this as:**

$$\begin{aligned} 0 &= A\, X_0 + B\, X_1 + C\, X_2, \\ 0 &= A'\, X_0 + B'\, X_1 + C'\, X_2. \end{aligned}$$

- **Form a** 2×2 **determinant:**

$$\begin{vmatrix} A\,X_0 & B\,X_1 \\ A'\,X_0 & B'\,X_1 \end{vmatrix}.$$

- **Replace the second column, and compute:**

$$\begin{aligned} \begin{vmatrix} A\,X_0 & B\,X_1 \\ A'\,X_0 & B'\,X_1 \end{vmatrix} &= \begin{vmatrix} A\,X_0 & \underline{-A\,X_0} - C\,X_2 \\ A'\,X_0 & \underline{-A'\,X_0} - C'\,X_2 \end{vmatrix} \\ &= -\begin{vmatrix} A\,X_0 & C\,X_2 \\ A'\,X_0 & C'\,X_2 \end{vmatrix}. \end{aligned}$$

Lemma. *From the two equations:*

$$\begin{aligned} 0 &= A\,X_0 + B\,X_1 + C\,X_2, \\ 0 &= A'\,X_0 + B'\,X_1 + C'\,X_2. \end{aligned}$$

it follows:

$$\begin{vmatrix} A\,X_0 & B\,X_1 \\ A'\,X_0 & B'\,X_1 \end{vmatrix} = -\begin{vmatrix} A\,X_0 & C\,X_2 \\ A'\,X_0 & C'\,X_2 \end{vmatrix} = \begin{vmatrix} B\,X_0 & C\,X_2 \\ B'\,X_0 & C'\,X_2 \end{vmatrix}.$$

- **Fermat-Cramer Rational Miracle:** Divide by:

$$\frac{1}{X_0\,X_1\,X_2},$$

obtain:

$$\frac{\begin{vmatrix} A\,\underline{X_0}_{\circ} & B\,\underline{X_1}_{\circ} \\ A'\,\underline{X_0}_{\circ} & B'\,\underline{X_1}_{\circ} \end{vmatrix}}{\underline{X_0}_{\circ}\,\underline{X_1}_{\circ}\,X_2} = -\frac{\begin{vmatrix} A\,\underline{X_0}_{\circ} & C\,\underline{X_2}_{\circ} \\ A'\,\underline{X_0}_{\circ} & C'\,\underline{X_2}_{\circ} \end{vmatrix}}{\underline{X_0}_{\circ}\,X_1\,\underline{X_2}_{\circ}} = \frac{\begin{vmatrix} B\,\underline{X_0}_{\circ} & C\,\underline{X_2}_{\circ} \\ B'\,\underline{X_0}_{\circ} & C'\,\underline{X_2}_{\circ} \end{vmatrix}}{\underline{X_0}_{\circ}\,X_1\,\underline{X_2}_{\circ}}.$$

and simplify:

$$\boxed{\frac{\begin{vmatrix} A & B \\ A' & B' \end{vmatrix}}{\mathbf{X_2}} = -\frac{\begin{vmatrix} A & C \\ A' & C' \end{vmatrix}}{\mathbf{X_1}} = \frac{\begin{vmatrix} B & C \\ B' & C' \end{vmatrix}}{\mathbf{X_0}}}.$$

Rationality

- **Natural integer numbers:**

$$\mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6},\ \mathbf{7},\ \mathbf{8},\ \mathbf{9},\ \mathbf{10},\ \mathbf{11},\ \mathbf{12},\ \mathbf{13},\ \ldots$$

Thesis. [M.] *Advanced mathematics depends upon archetypical rational computational phenomena.*

- **Today:**

Exhibit Several Instances of Rationality

☐ Cartan theory.

☐ Complex Algebraic Geometry.

- **Three open sets:**

$$\{R_x \neq 0\},$$
$$\{R_y \neq 0\},$$
$$\{R_z \neq 0\}.$$

- **First strategy:**

$$\frac{\text{Jet differential}}{\mathbf{R_x}} = \frac{\text{Transferred jet differential}}{\mathbf{R_y}}.$$

- **Question 1:** Does this exist?
- **Second strategy from Čech cohomology:**

$$\frac{\text{Jet}}{\mathbf{R_x R_y}} = \frac{\text{Jet}}{\mathbf{R_z R_x}} = \frac{\text{Jet}}{\mathbf{R_y R_z}}.$$

- **Only codimension 2 singularities remain:**

$$\{0 = R_x = R_y\} \bigcup \{0 = R_z = R_x\} \bigcup \{0 = R_y = R_z\}.$$

- **Question 2:** Does this exist?
- **Third strategy:**

$$\frac{\text{Jet differential}}{\mathbf{R_x}} = \frac{\text{jet differential}}{\mathbf{R_y}} + \frac{\text{jet differential}}{\mathbf{R_z}}.$$

- **Question 3:** Does this exist?

Explicit Chern-Moser Tensors

• **Real hypersurface:** $M^{2n+1} \subset \mathbb{C}^{n+1}$, Levi-nondegenerate, real-analytic ($\mathscr{C}^\omega$).

• **General** $n \geqslant 2$**:** Coordinates:

$$(z_1, \dots, z_n, w).$$

• **Implicit defining equation:**

$$\rho\big(z_1, \dots, z_n, w, \overline{z}_1, \dots, \overline{z}_n, \overline{w}\big) = 0.$$

Theorem. [Chern-Moser 1974] *The primary curvature tensor:*

$$\Phi_\alpha^\beta = S_{\alpha\rho}^{\beta\sigma}\, \omega^\rho \wedge \omega_\sigma + \text{normalized remainder},$$

appears in:

$$d\varphi_\alpha^\beta = \Phi_\alpha^\beta + \text{normalized remainder}.$$

where $1 \leqslant \alpha, \rho, \beta, \sigma \leqslant n$*.*

• **Absent from the literature:** *Compute explicitly the* $S_{\alpha\rho}^{\beta\sigma}$ *for* $n \geqslant 2$*.*

Despite their importance, until now, the invariants of pseudoconvex domains have been fully completed, to our knowledge, only in the case of the unit ball $\mathbb{B}^{n+1} \subset \mathbb{C}^{n+1}$, where they all vanish! Webster 2000

- **Hypersurface:**

$$M = \{\rho = 0\}.$$

- **Smooth:**

$$d\rho(p) \neq 0 \qquad (\forall\, p \in M).$$

- **Levi nondegeneracy:**

$$\mathsf{L}(\rho)(p) \neq 0 \qquad (\forall\, p \in M),$$

where:

$$\mathsf{L}(\rho) := \det \begin{pmatrix} 0 & \rho_{z_1} & \cdots & \rho_{z_n} & \rho_w \\ \rho_{\overline{z}_1} & \rho_{z_1\overline{z}_1} & \cdots & \rho_{z_n\overline{z}_1} & \rho_{w\overline{z}_1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{\overline{z}_n} & \rho_{z_1\overline{z}_n} & \cdots & \rho_{z_n\overline{z}_n} & \rho_{w\overline{z}_n} \\ \rho_{\overline{w}} & \rho_{z_1\overline{w}} & \cdots & \rho_{z_n\overline{w}} & \rho_{w\overline{w}} \end{pmatrix}.$$

- **Example:** $n = 2$:

$$\mathsf{L}(\rho) \;:=\; \mathsf{det} \begin{pmatrix} 0 & \rho_{z_1} & \rho_{z_2} & \rho_w \\ \rho_{\overline{z}_1} & \rho_{z_1\overline{z}_1} & \rho_{z_2\overline{z}_1} & \rho_{w\overline{z}_1} \\ \rho_{\overline{z}_2} & \rho_{z_1\overline{z}_2} & \rho_{z_2\overline{z}_2} & \rho_{w\overline{z}_2} \\ \rho_{\overline{w}} & \rho_{z_1\overline{w}} & \rho_{z_2\overline{w}} & \rho_{w\overline{w}} \end{pmatrix}.$$

- **Introduce:** For all $1 \leqslant i \leqslant n$ and all $1 \leqslant \ell \leqslant n+1$:

$$\mathsf{L}_{i,\ell}(\rho) \;:=\; \mathsf{det} \begin{pmatrix} \text{delete row } \; i+1 \\ \text{delete column } \; \ell+1 \end{pmatrix}.$$

- **Example:** $n = 2$:

$$\begin{vmatrix} 0 & \rho_{z_1} & \rho_{z_2} & \rho_w \\ \rho_{\overline{z}_1} & \rho_{z_1\overline{z}_1} & \rho_{z_2\overline{z}_1} & \rho_{w\overline{z}_1} \\ \rho_{\overline{z}_2} & \rho_{z_1\overline{z}_2} & \rho_{z_2\overline{z}_2} & \rho_{w\overline{z}_2} \\ \rho_{\overline{w}} & \rho_{z_1\overline{w}} & \rho_{z_2\overline{w}} & \rho_{w\overline{w}} \end{vmatrix} \qquad \longmapsto \qquad \begin{vmatrix} 0 & \rho_{z_2} & \rho_w \\ \rho_{\overline{z}_2} & \rho_{z_2\overline{z}_2} & \rho_{w\overline{z}_2} \\ \rho_{\overline{w}} & \rho_{z_2\overline{w}} & \rho_{w\overline{w}} \end{vmatrix}.$$

- **Introduce:** Derivations:

$$\mathsf{D}_1 \;:=\; \rho_w\bigg(-\frac{\mathsf{L}_{1,1}}{\mathsf{L}}\frac{\partial}{\partial\overline{z}_1}+\cdots+(-1)^n\frac{\mathsf{L}_{1,n}}{\mathsf{L}}\frac{\partial}{\partial\overline{z}_n}+(-1)^{n+1}\frac{\mathsf{L}_{1,n+1}}{\mathsf{L}}\frac{\partial}{\partial\overline{w}}\bigg),$$

....

$$\mathsf{D}_n \;:=\; \rho_w\bigg(-\frac{\mathsf{L}_{n,1}}{\mathsf{L}}\frac{\partial}{\partial\overline{z}_1}+\cdots+(-1)^n\frac{\mathsf{L}_{n,n}}{\mathsf{L}}\frac{\partial}{\partial\overline{z}_n}+(-1)^{n+1}\frac{\mathsf{L}_{n,n+1}}{\mathsf{L}}\frac{\partial}{\partial\overline{w}}\bigg),$$

- **Example:** $n=2$:

$$\mathsf{D}_1 \;:=\; \rho_w\left(-\frac{\begin{vmatrix}0&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{z}_1}+\frac{\begin{vmatrix}0&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{z}_2}-\frac{\begin{vmatrix}0&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{w}}\right),$$

$$\mathsf{D}_2 \;:=\; \rho_w\left(\frac{\begin{vmatrix}0&\rho_{z_1}&\rho_w\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{z}_1}-\frac{\begin{vmatrix}0&\rho_{z_1}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{z}_2}+\frac{\begin{vmatrix}0&\rho_{z_1}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{w\overline{z}_2}\end{vmatrix}}{\begin{vmatrix}0&\rho_{z_1}&\rho_{z_2}&\rho_w\\ \rho_{\overline{z}_1}&\rho_{z_1\overline{z}_1}&\rho_{z_2\overline{z}_1}&\rho_{w\overline{z}_1}\\ \rho_{\overline{z}_2}&\rho_{z_1\overline{z}_2}&\rho_{z_2\overline{z}_2}&\rho_{w\overline{z}_2}\\ \rho_{\overline{w}}&\rho_{z_1\overline{w}}&\rho_{z_2\overline{w}}&\rho_{w\overline{w}}\end{vmatrix}}\frac{\partial}{\partial\overline{w}}\right).$$

- **Introduce:** For $1 \leqslant i, j \leqslant n$:

$$\mathsf{H}_{i,j}(\rho) := \frac{-\rho_{z_i}\rho_{z_j}\rho_{ww} + \rho_{z_i}\rho_w\rho_{z_j w} + \rho_{z_j}\rho_w\rho_{z_i w} - \rho_w\rho_w\rho_{z_i z_j}}{\rho_w\,\rho_w\,\rho_w}.$$

Theorem. [Foo Ph.D. Orsay 2017] *For $n = 2$, the CR-umbilical locus is the zero-set of* **5** *equations:*

$$0 = \mathsf{D}_1\Big(\mathsf{D}_1\big(\mathsf{H}_{2,2}(\rho)\big)\Big),$$

$$0 = \mathsf{D}_1\Big(\mathsf{D}_1\big(\mathsf{H}_{1,2}(\rho)\big)\Big) - \mathsf{D}_1\Big(\mathsf{D}_2\big(\mathsf{H}_{2,2}(\rho)\big)\Big),$$

$$0 = \mathsf{D}_1\Big(\mathsf{D}_1\big(\mathsf{H}_{1,1}(\rho)\big)\Big) - 4\,\mathsf{D}_1\Big(\mathsf{D}_2\big(\mathsf{H}_{1,2}(\rho)\big)\Big) + \mathsf{D}_2\Big(\mathsf{D}_2\big(\mathsf{H}_{2,2}(\rho)\big)\Big)$$

$$0 = \mathsf{D}_2\Big(\mathsf{D}_2\big(\mathsf{H}_{2,1}(\rho)\big)\Big) - \mathsf{D}_2\Big(\mathsf{D}_1\big(\mathsf{H}_{1,1}(\rho)\big)\Big),$$

$$0 = \mathsf{D}_2\Big(\mathsf{D}_2\big(\mathsf{H}_{1,1}(\rho)\big)\Big).$$

Theorem. [Foo, Ph.D. Orsay 2017] *For $n \geqslant 2$, the CR-umbilical locus is the zero-set:*

$$
\begin{aligned}
0 = \; & \mathsf{D}_{i_1}\Big(\mathsf{D}_{i_2}\big(\mathsf{H}_{k_1,k_2}\big)\Big) - \\
& - \frac{1}{n+2} \sum_{\ell=1}^{n} \bigg(\delta_{k_1,i_1}\,\mathsf{D}_\ell\Big(\mathsf{D}_{i_2}\big(\mathsf{H}_{\ell,k_2}\big)\Big) + \delta_{k_1,i_2}\,\mathsf{D}_{i_1}\Big(\mathsf{D}_\ell\big(\mathsf{H}_{\ell,k_2}\big)\Big) + \\
& \qquad + \delta_{k_2,i_1}\,\mathsf{D}_\ell\Big(\mathsf{D}_{i_2}\big(H_{k_1,\ell}\big)\Big) + \delta_{k_2,i_2}\,\mathsf{D}_{i_1}\Big(\mathsf{D}_\ell\big(H_{k_1,\ell}\big)\Big) + \\
& + \frac{1}{(n+1)(n+2)}\Big[\delta_{k_1,i_1}\,\delta_{k_2,i_2} + \delta_{k_2,i_1}\,\delta_{k_1,i_2}\Big] \sum_{\ell_1=1}^{n}\sum_{\ell_2=1}^{n} \mathsf{D}_{\ell_1}\Big(\mathsf{D}_{\ell_2}\big(\mathsf{H}_{\ell_1,\ell_2}\big)\Big),
\end{aligned}
$$

for all:

$$
\begin{aligned}
1 \leqslant\; k_1,\, k_2 \;\leqslant\; n, \\
1 \leqslant\; i_1,\, i_2 \;\leqslant\; n.
\end{aligned}
$$

Nilpotent Lie Algebras and CR Models up to Dimension 5

• Real analytic CR manifold:

□ Integers $n \geqslant 1$ and $d \geqslant 1$.

□ Complex space $\mathbb{C}^{n+d}$.

□ Real submanifold $M^{2n+d} \subset \mathbb{C}^{n+d}$ with tangent planes at $p \in M$:

$$\begin{aligned} T_pM &\cong \mathbb{C}^n \times \mathbb{R}^d \\ &\subset \mathbb{C}^n \times \mathbb{C}^d. \end{aligned}$$

□ Coordinates:

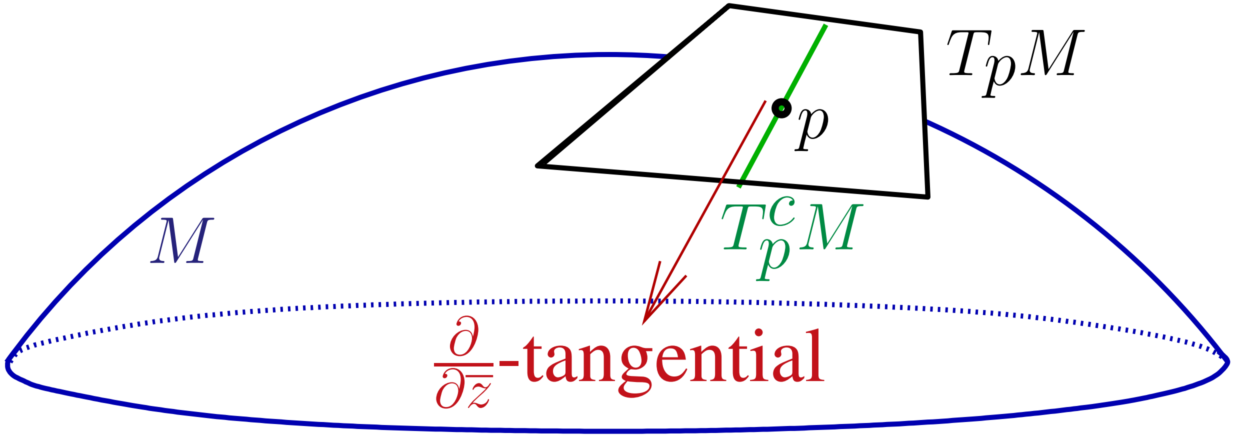
$$\big(z_1, \dots, z_k, \dots, z_{n+d}\big) \qquad \text{avec} \quad z_k = x_k + \sqrt{-1}\, y_k.$$

□ Complex structure of $T\mathbb{C}^{n+d}$:

$$J\bigg(\frac{\partial}{\partial x_k}\bigg) := \frac{\partial}{\partial y_k} \qquad \text{et} \qquad J\bigg(\frac{\partial}{\partial y_k}\bigg) := -\frac{\partial}{\partial x_k}.$$

□ Complex tangent space invariant by $J \equiv \times\sqrt{-1}$:

$$T^cM := TM \cap J(TM).$$



- **So:**

$$M^{2n+d} \subset \mathbb{C}^{n+d}.$$

- **With:**

$$T^cM = TM \cap JTM.$$

- **Iterated Lie brackets:**

$$\begin{aligned}
D^{-1}M &:= T^cM, \\
D^{-2}M &:= \mathsf{Vect}_{\mathscr{C}^\omega}\big(D^{-1}M + \big[T^cM,\, D^{-1}M\big]\big), \\
D^{-3}M &:= \mathsf{Vect}_{\mathscr{C}^\omega}\big(D^{-2}M + \big[T^cM,\, D^{-2}M\big]\big), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
D^{-k-1}M &:= \mathsf{Vect}_{\mathscr{C}^\omega}\big(D^{-k}M + \big[T^cM,\, D^{-k}M\big]\big).
\end{aligned}$$

Lemme. [Immediate] *Growth vectors* $n = 1$*:*

dimension **3** : $\big(\mathbf{2}, \mathbf{1}\big)$;

dimension **4** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{1}\big)$;

dimension **5** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{2}\big)$;

dimension **5** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}\big)$. $\square$

Classification of nilpotent Lie algebras in dimension $\leqslant 5$

[*cf.* Goze-Remm]

Dimension **1**:

$$\mathfrak{a}_1 := \mathbb{R}.$$

Dimension **2**:

$$\mathfrak{a}_2 = \mathfrak{a}_1 \oplus \mathfrak{a}_1$$

Dimension **3**: The decomposable:

$$\mathfrak{a}_3 := \mathfrak{a}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{a}_1,$$

and the *Heisenberg*:

$$\boxed{\mathfrak{n}_3^1 : \qquad [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3}\,.$$

Dimension **4**: The two decomposable:

$$\mathfrak{a}_1^{\oplus 4} \qquad \text{and} \qquad \mathfrak{a}_1 \oplus \mathfrak{n}_1^3,$$

and:

$$\boxed{\mathfrak{n}_4^1 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4. \end{cases}}$$

Dimension **5**: The three decomposable:

$$\mathfrak{a}_1^{\oplus 5}, \qquad \mathfrak{n}_3^1 \oplus \mathfrak{a}_2, \qquad \mathfrak{n}_4^1 \oplus \mathfrak{a}_1,$$

Dimension 5, irreducible: *There exist* **6** *non-isomorphic real nilpotent Lie algebras:*

☐ $c(\mathfrak{g}) = (4,1)$ (filiform case):

$$\mathfrak{n}_5^1: \quad \begin{cases} [x_1, x_2] = x_3 \\ [x_1, x_3] = x_4 \\ [x_1, x_4] = x_5 \end{cases}$$

$$\mathfrak{n}_5^2: \quad \begin{cases} [x_1, x_2] = x_3 \\ [x_1, x_3] = x_4 \\ [x_1, x_4] = x_5 \\ [x_2, x_3] = x_5 \end{cases}$$

☐ $c(\mathfrak{g}) = (3,1,1)$:

$$\mathfrak{n}_5^3: \quad \begin{cases} [x_1, x_2] = x_3 \\ [x_1, x_3] = x_4 \\ [x_2, x_5] = x_4 \end{cases}$$

$$\mathfrak{n}_5^4: \quad \begin{cases} [x_1, x_2] = x_3 \\ [x_1, x_3] = x_4 \\ [x_2, x_3] = x_5 \end{cases}$$

☐ $c(\mathfrak{g}) = (2,2,1)$:

$$\mathfrak{n}_5^5: \quad \begin{cases} [x_1, x_2] = x_3 \\ [x_1, x_4] = x_5 \end{cases}$$

☐ $c(\mathfrak{g}) = (2, 1, 1, 1)$:

$$\mathfrak{n}_5^5 : \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3 \\ [\mathsf{x}_4, \mathsf{x}_5] = \mathsf{x}_3 \end{cases}$$

- **Classification:** Dimension **6**:

[Vergne, Seeley, Carles, Goze-Khakimdjanov].

- **Classification:** Dimension **7**:

[Goze-Remm].

- **Several branches:** Dimensions **8**, **9**:

[Goze-Remm].

- **Associated equivalences of CR manifolds!**

Models of CR submanifolds in dimension $\leqslant$ 5

- **Dimension:**

$$2n + d \qquad\qquad (n \geqslant 1 \textbf{ et } d \geqslant 1).$$

- **Dimension 3:**

$$n = 1, \qquad\qquad d = 1.$$

Levi non-degenerate hypersurface:

$$M^3 \subset \mathbb{C}^2.$$

- **Dimension 4:** [Beloshapka-Ezhov-Schmalz 2007]

$$n = 1, \qquad\qquad d = 2.$$

- **Dimension 5:**

$$n = 2, \qquad\qquad d = 1,$$

$$n = 1, \qquad\qquad d = 3.$$

Lemme. [Recall] *In CR dimension $n = 1$, possible growth vectors are:*

dimension **3** : $\big(\mathbf{2}, \mathbf{1}\big)$;

dimension **4** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{1}\big)$;

dimension **5** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{2}\big)$;

dimension **5** : $\big(\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{1}\big)$. □

- **Equivalences of hypersurfaces $M^3 \subset \mathbb{C}^2$:**

Proposition. [Easy] *Every hypersurface $M^3 \subset \mathbb{C}^2$ satisfying:*

$$D^1 M = T^c M \quad \text{is of rank } 2,$$

$$D^2 M = T^c M + [T^c M, T^c M] \quad \text{is of rank } 3,$$

can be represented in coordinates $(z, w) \in \mathbb{C}^2$, by:

$$w - \overline{w} = 2i\, z\overline{z} + \mathrm{O}_{\mathsf{poids}}(3).$$

Proposition. [Beloshapka 1997] *Every CR submanifold $M^5 \subset \mathbb{C}^4$ of codimension $d = 3$ with:*

$$D^1 M = T^c M \quad \text{is of rank } 2,$$

$$D^2 M = T^c M + [T^c M, T^c M] \quad \text{is of rank } 3,$$

$$D^3 M = T^c M + [T^c M, T^c M] + \big[T^c M, [T^c M, T^c M]\big]$$

if of maximal possible rank 5,

can be represented as:

$$\left[\begin{aligned} w_1 - \overline{w}_1 &= 2i\, z\overline{z} && + \mathrm{O}_{\mathsf{poids}}(4) \\ w_2 - \overline{w}_2 &= 2i\, z\overline{z}(z + \overline{z}) && + \mathrm{O}_{\mathsf{poids}}(4). \\ w_3 - \overline{w}_3 &= 2\, z\overline{z}(z - \overline{z}) && + \mathrm{O}_{\mathsf{poids}}(4). \end{aligned}\right.$$

- **Real-analytic:**

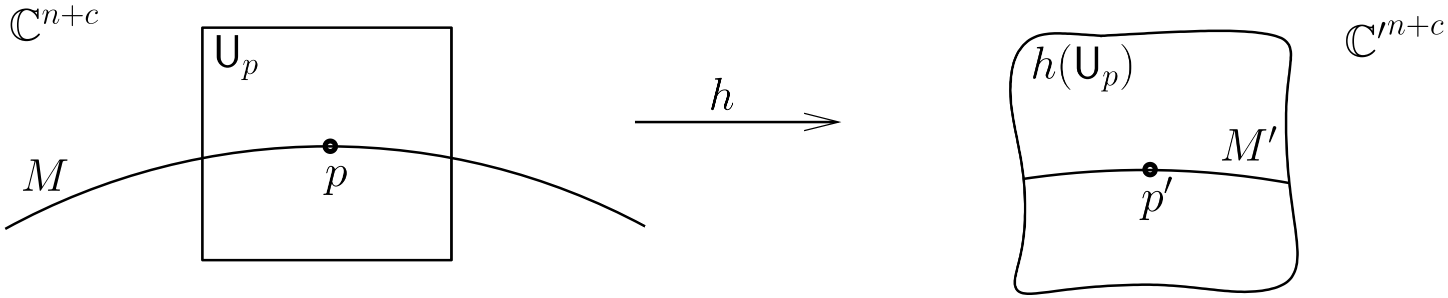
$$M \in \mathscr{C}^{\omega}.$$

- **Problem.** *Classify the* $\mathscr{C}^{\omega}$*:*

$$M^{2n+c} \subset \mathbb{C}^{n+c}$$

modulo local biholomorphisms of $\mathbb{C}^{n+c}$ *up to dimension:*

$$2n + c \leqslant \mathbf{5}.$$



- **Possibilities:**

$$2n + c = \mathbf{3} \quad \Longrightarrow \quad \big\{ n = \mathbf{1}, \quad c = \mathbf{1},$$

$$2n + c = \mathbf{4} \quad \Longrightarrow \quad \big\{ n = \mathbf{1}, \quad c = \mathbf{2},$$

$$2n + c = \mathbf{5} \quad \Longrightarrow \quad \begin{cases} n = \mathbf{1}, & c = \mathbf{3}, \\ n = \mathbf{2}, & c = \mathbf{1}. \end{cases}$$

- **Coordinates:**

$$\big(z_1, \dots, z_n, w_1, \dots, w_c\big) = \big(x_1 + \sqrt{-1}\, y_1, \dots, x_n + \sqrt{-1}\, y_n,\\ u_1 + \sqrt{-1}\, v_1, \dots, u_c + \sqrt{-1}\, v_c\big),$$

- **Graphing functions:**

$$M^3 \subset \mathbb{C}^2\colon \qquad \Big[\; v = \varphi(x, y, u),$$

$$M^4 \subset \mathbb{C}^3\colon \qquad \left[\begin{array}{l} v_1 = \varphi_1(x, y, u_1, u_2), \\ v_2 = \varphi_2(x, y, u_1, u_2), \end{array}\right.$$

$$M^5 \subset \mathbb{C}^4\colon \qquad \left[\begin{array}{l} v_1 = \varphi_1(x, y, u_1, u_2, u_3), \\ v_2 = \varphi_2(x, y, u_1, u_2, u_3), \\ v_3 = \varphi_3(x, y, u_1, u_2, u_3), \end{array}\right.$$

$$M^5 \subset \mathbb{C}^3\colon \qquad \Big[\; v = \varphi(x_1, y_1, x_2, y_2, u).$$

- **Fundamental bundle:**

$$T^{1,0}M := \Big\{X - \sqrt{-1}\, J(X)\colon\, X \in \Gamma(T^c M)\Big\}.$$

- **Integrability:**

$$\big[T^{1,0}M,\, T^{1,0}M\big] \subset T^{1,0}M.$$

- **Conjugate:**

$$T^{0,1}M := \overline{T^{1,0}M}.$$

- **Non-holonomic:**

$$\big[T^{1,0}M,\, T^{0,1}M\big] \not\subset T^{1,0}M \oplus T^{0,1}M.$$

- **Local generators of $T^{1,0}M$:**

$$\mathscr{L}_1, \dots, \mathscr{L}_n.$$

Theorem. *Excluding degenerate cases, there are precisely* **6** *classes of $\mathscr{C}^\omega$ CR submanifolds $M^{2n+c} \subset \mathbb{C}^{n+c}$ of dimension:*

$$2n + c \leqslant \mathbf{5},$$

hence of CR dimension $n = 1$ ou $n = 2$, with:

$$\{\mathscr{L}\} \qquad \text{or} \qquad \{\mathscr{L}_1, \mathscr{L}_2\},$$

being generators of $T^{1,0}M$.

- **Classe I:** Hypersurfaces $M^3 \subset \mathbb{C}^2$ such that $\big\{\mathscr{L},\, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:

Model I: $\qquad v = z\bar{z}.$

• **Class** II**:** CR-generic $M^4 \subset \mathbb{C}^3$ such that $\{\mathscr{L}, \overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}], [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:

$$\text{Model II:} \qquad \begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2. \end{cases}$$

• **Class** III$_1$**:** CR-generic $M^5 \subset \mathbb{C}^4$ such that $\{\mathscr{L}, \overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}], [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:

$$\text{Model III}_1\text{:} \qquad \begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2, \\ v_3 = \sqrt{-1}\left(z^2\overline{z} - z\overline{z}^2\right). \end{cases}$$

• **Class** III$_2$**:** CR-generic $M^5 \subset \mathbb{C}^4$ such that $\{\mathscr{L}, \overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}], [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], [\mathscr{L}, [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]]]\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, while $\mathbf{4} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}, \overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}], [\mathscr{L}, [\mathscr{L}, \overline{\mathscr{L}}]], [\overline{\mathscr{L}}, [\mathscr{L}, \overline{\mathscr{L}}]]\Big)$, with:

$$\text{Model III}_2\text{:} \qquad \begin{cases} v_1 = z\overline{z}, \\ v_2 = z^2\overline{z} + z\overline{z}^2, \\ v_3 = 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2. \end{cases}$$

• **Class IV$_1$:** Hypersurfaces $M^5 \subset \mathbb{C}^3$ such that $\big\{\mathscr{L}_1,\, \mathscr{L}_2,\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{L}}_2,\, \big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, and such that the Levi form of M has rank 2 at every point $p \in M$, with:

$$\text{Model(s) IV}_1\text{:} \qquad v = z_1\overline{z}_1 \pm z_2\overline{z}_2.$$

• **Class IV$_2$:** Hypersurfaces $M^5 \subset \mathbb{C}^3$ such that $\big\{\mathscr{L}_1,\, \mathscr{L}_2,\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{L}}_2,\, \big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big\}$ is a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, such that the Levi form is of rank 1 at every point $p \in M$, while the Freeman form is nondegenerate, with:

$$\text{Model IV}_2\text{:} \qquad v = \frac{z_1\overline{z}_1 + \frac{1}{2}\, z_1z_1\overline{z}_2 + \frac{1}{2}\, z_2\overline{z}_1\overline{z}_1}{1 - z_2\overline{z}_2}.$$

• **Existence of the 6 classes I, II, III$_1$, III$_2$, IV$_1$, IV$_2$:** Graphing functions are arbitrary.

Proposition. *For the six classes* I, II, III$_1$, IV$_1$*:*

$$\underline{\text{(I)}}\text{:} \qquad \big[\, v = z\overline{z} + z\overline{z}\, \mathrm{O}_1\big(z, \overline{z}\big) + z\overline{z}\, \mathrm{O}_1(u),$$

$$\underline{\text{(II)}}\text{:} \qquad \begin{cases} v_1 = z\overline{z} + z\overline{z}\, \mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\, \mathrm{O}_1(u_1) + z\overline{z}\, \mathrm{O}_1(u_2), \\ v_2 = z^2\overline{z} + z\overline{z}^2 + z\overline{z}\, \mathrm{O}_2\big(z, \overline{z}\big) + z\overline{z}\, \mathrm{O}_1(u_1) + z\overline{z}\, \mathrm{O}_1(u_2), \end{cases}$$

$$\underline{(\mathrm{III})_1}:\ \left[\begin{aligned} v_1 &= z\overline{z} && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \\ v_3 &= \sqrt{-1}\,\big(z^2\overline{z} - z\overline{z}^2\big) && + z\overline{z}\,\mathrm{O}_2\big(z,\overline{z}\big) + z\overline{z}\,\mathrm{O}_1(u_1) + z\overline{z}\,\mathrm{O}_1(u_2) + z\overline{z}\,\mathrm{O}_1(u_3), \end{aligned}\right.$$

$$\underline{(\mathrm{IV})_1}:\qquad \Big[\, v = z_1\overline{z}_1 \pm z_2\overline{z}_2 + \mathrm{O}_3\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big),$$

with arbitrary remainders. For the class:

$$\underline{(\mathrm{III})_2}:\ \begin{aligned} v_1 &= z\overline{z} + c_1\, z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_2 &= z^2\overline{z} + z\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \\ v_3 &= 2\,z^3\overline{z} + 2\,z\overline{z}^3 + 3\,z^2\overline{z}^2 + z\overline{z}\,\mathrm{O}_3\big(z,\overline{z}\big) + z\overline{z}\,u_1\,\mathrm{O}_1\big(z,\overline{z},u_1\big) + \\ &\qquad + z\overline{z}\,u_2\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2\big) + z\overline{z}\,u_3\,\mathrm{O}_1\big(z,\overline{z},u_1,u_2,u_3\big), \end{aligned}$$

the 3 *graphing functions* φ_1, φ_2, φ_3 *must satisfy:*

$$0 \equiv \begin{vmatrix} \mathscr{L}(\overline{A}_1)-\overline{\mathscr{L}}(A_1) & \mathscr{L}(\overline{A}_2)-\overline{\mathscr{L}}(A_2) & \mathscr{L}(\overline{A}_3)-\overline{\mathscr{L}}(A_3) \\ \begin{matrix}\mathscr{L}(\mathscr{L}(\overline{A}_1))-2\mathscr{L}(\overline{\mathscr{L}}(A_1))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_1))\end{matrix} & \begin{matrix}\mathscr{L}(\mathscr{L}(\overline{A}_2))-2\mathscr{L}(\overline{\mathscr{L}}(A_2))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_2))\end{matrix} & \begin{matrix}\mathscr{L}(\mathscr{L}(\overline{A}_3))-2\mathscr{L}(\overline{\mathscr{L}}(A_3))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_3))\end{matrix} \\ \begin{matrix}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_1))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_1))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_1))\end{matrix} & \begin{matrix}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_2))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_2))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_2))\end{matrix} & \begin{matrix}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_3))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_3))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_3))\end{matrix} \end{vmatrix}.$$

Lastly, for the class:

$$\underline{(\mathrm{IV})_2}:\quad \left[\begin{aligned} v = {} & z_1\overline{z}_1 + \tfrac{1}{2}\,z_1z_1\overline{z}_2 + \tfrac{1}{2}\,z_2\overline{z}_1\overline{z}_1 + \\ & + \mathrm{O}_4\big(z_1,z_2,\overline{z}_1,\overline{z}_2\big) + u\,\mathrm{O}_2\big(z_1,z_2,\overline{z}_1,\overline{z}_2,u\big), \end{aligned}\right.$$

the graphing function φ *has zero Levi determinant.*

Results

Theorem. [Merker-Sabzevari 2013] *For a Levi nondegenerate hypersurface* $M^3 \subset \mathbb{C}^2$ *graphed as:*

$$v = \varphi(x, y, u),$$

the single Cartan curvature which depends on the 6*-th order jet* $J^6_{x,y,u}\varphi$ *of the graphing function has:*

$$\sim \mathbf{1\,500\,000 \ terms}.$$

- **Six classes:**

$$\mathsf{I}, \quad \mathsf{II}, \quad \mathsf{III}_1, \quad \mathsf{III}_2, \quad \mathsf{IV}_1, \quad \mathsf{IV}_2.$$

Theorem. [Merker-Pocchiola-Sabzevari] Explicit reductions to an absolute parallelism or to a Cartan connection for all six classes of CR manifolds at a generic point up to dimension:

$$\mathbf{3}, \quad \mathbf{4}, \quad \mathbf{5}.$$

Theorem. [Merker-Pocchiola-Sabzevari] Curvatures explicit in terms of graphing functions.

• **Class** IV_2**:**

• **Isaev-Zaitsev 2013 and Medori-Spiro 2013:** Existence of an absolute parallelism on a certain 10-dimensional manifold P related to the biholomorphic equivalence problem between holomorphically nondegenerate hypersurfaces having Levi form of constant rank 1.

Theorem. [Merker-Pocchiola-Sabzevari] Explicit reductions to an absolute parallelism or to a Cartan connection for all six classes of CR manifolds at a generic point up to dimension:

$$\mathbf{3}, \quad \mathbf{4}, \quad \mathbf{5}.$$

Theorem. [Pocchiola, Ph.D. 2014 Orsay] *Precisely* **2** *invariants:*

$$\mathbf{J} \qquad \textit{and} \qquad \mathbf{W}$$

are related to the biholomorphic equivalence problem for 2*-nondegenerate hypersurfaces* $M^5 \subset \mathbb{C}^3$ *having degenerate Levi form of constant rank* 1*.*

• **Isaev 2014:** Recovering of Pocchiola's two invariants in the special tube case:

$$v = \varphi(x_1, x_2).$$

• **Fact:** Pocchiola's formulas are invariant, they have exactly the same form in the general case:

$$v = \varphi\big(x_1, y_1, x_2, y_2, u\big).$$

• **Isaev 2014:** If a tube hypersurface is locally biholomorphic to the light cone, then it is in fact affinely biholomorphic to it.

• **Merker 2014:** Direct short proof without using **J** and **W** (unpublished).

Hypersurfaces $M^5 \subset \mathbb{C}^3$ of Levi-Rank 1

[Ph.D. Pocchiola 2014]

- **Local real analytic hypersurface:**

$$M^5 \subset \mathbb{C}^3.$$

- **Graph:**

$$\begin{aligned} u &= F(z_1, z_2, \overline{z}_1, \overline{z}_2, v) \\ &= z_1\overline{z}_1 + \tfrac{1}{2}\, z_1^2\, \overline{z}_2 + \tfrac{1}{2}\, \overline{z}_1^2 z_2 + z_1\overline{z}_1 z_2\overline{z}_2 + \mathrm{O}(5). \end{aligned}$$

- **Two generators of $T^{1,0}M$:**

$$\mathscr{L}_1 = \frac{\partial}{\partial z_1} + A_1\,\frac{\partial}{\partial w},$$
$$\mathscr{L}_2 = \frac{\partial}{\partial z_2} + A_2\,\frac{\partial}{\partial w}.$$

- **Assume Levi form is of contant rank 1:**

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \sqrt{-1}\,\big(\mathscr{L}_1(\overline{A}^1) - \overline{\mathscr{L}_1}(A^1)\big) & \sqrt{-1}\,\big(\mathscr{L}_2(\overline{A}^1) - \overline{\mathscr{L}_1}(A^2)\big) \\ \sqrt{-1}\,\big(\mathscr{L}_1(\overline{A}^2) - \overline{\mathscr{L}_2}(A^1)\big) & \sqrt{-1}\,\big(\mathscr{L}_2(\overline{A}^2) - \overline{\mathscr{L}_2}(A^2)\big) \end{pmatrix} \begin{pmatrix} k \\ 1 \end{pmatrix}.$$

- **Generator of the kernel of the Levi form:**

$$\boxed{\mathscr{K} := \mathscr{L}_2 + k\,\mathscr{L}_1.}$$

- **Function-Coefficient:**

$$\boxed{k := \frac{-\mathscr{L}_2\big(\overline{A}^1\big) + \overline{\mathscr{L}_1}\big(A^2\big)}{\mathscr{L}_1\big(\overline{A}^1\big) - \overline{\mathscr{L}_1}\big(A^1\big)} = -\,\overline{z_1} + \mathrm{O}(2)}\,.$$

- **CR-transversal field:**

$$\mathscr{T} := i\,\big[\mathscr{L}_1, \overline{\mathscr{L}_1}\big].$$

- **Bracket:**

$$\begin{aligned}
\big[\mathscr{K}, \overline{\mathscr{L}_1}\big] &= \big[k\,\mathscr{L}_1 + \mathscr{L}_2, \overline{\mathscr{L}_1}\big] \\
&= k\,\big[\mathscr{L}_1, \overline{\mathscr{L}_1}\big] + \big[\mathscr{L}, \overline{\mathscr{L}_1}\big] - \overline{\mathscr{L}_1}(k)\,\mathscr{L}_1 \\
&= -\overline{\mathscr{L}_1}(k)\,\mathscr{L}_1.
\end{aligned}$$

• Lie structure:

$$
\begin{aligned}
\left[\mathscr{T},\mathscr{L}\right] &= -P\,\mathscr{T}, \\
\left[\mathscr{T},\overline{\mathscr{L}}\right] &= -\overline{P}\,\mathscr{T}, \\
\left[\mathscr{T},\mathscr{K}\right] &= \mathscr{L}_1(k)\,\mathscr{T}+\mathscr{T}(k)\,\mathscr{L}_1, \\
\left[\mathscr{T},\overline{\mathscr{K}}\right] &= \overline{\mathscr{L}_1}(\overline{k})\,\mathscr{T}+\mathscr{T}(\overline{k})\,\overline{\mathscr{L}_1}, \\
\left[\mathscr{L}_1,\overline{\mathscr{L}_1}\right] &= -i\,\mathscr{T}, \\
\left[\mathscr{L}_1,\mathscr{K}\right] &= \mathscr{L}_1(k)\,\mathscr{L}_1, \\
\left[\mathscr{L}_1,\overline{\mathscr{K}}\right] &= \mathscr{L}_1(\overline{k})\,\overline{\mathscr{L}_1}, \\
\left[\overline{\mathscr{L}_1},\mathscr{K}\right] &= \overline{\mathscr{L}_1}(k)\,\mathscr{L}_1 \\
\left[\overline{\mathscr{L}_1},\overline{\mathscr{K}}\right] &= \overline{\mathscr{L}_1}(\overline{k})\,\overline{\mathscr{L}_1} \\
\left[\mathscr{K},\overline{\mathscr{K}}\right] &= 0.
\end{aligned}
$$

• Jacobi relations:

$$\mathscr{K}(P) = -P\,\mathscr{L}_1(k) - \mathscr{L}_1\left(\mathscr{L}_1(k)\right),$$

and

$$\mathscr{K}(\overline{P}) = -P\,\overline{\mathscr{L}_1}(k) - \mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right).$$

• Darboux-Cartan structure of the initial coframe:

$$\begin{cases} d\rho_0 = P\ \rho_0 \wedge \kappa_0 + \overline{P}\ \rho_0 \wedge \overline{\kappa}_0 - \overline{\mathscr{L}_1}(k)\ \rho_0 \wedge \zeta_0 - \overline{\mathscr{L}_1}(\overline{k})\ \rho_0 \wedge \overline{\zeta_0} + i\ \kappa \wedge \kappa_0 \\ d\kappa_0 = -\mathscr{L}_1(k)\ \kappa_0 \wedge \zeta_0 + \overline{\mathscr{L}_1}(k)\ \zeta_0 \wedge \overline{\kappa}_0 - \mathscr{T}(k)\ \rho_0 \wedge \kappa_0 \\ d\zeta_0 = 0 \\ d\overline{\kappa}_0 = -\overline{\mathscr{L}_1}(\overline{k})\ \overline{\kappa_0} \wedge \overline{\zeta_0} - \mathscr{L}_1(k)\ \kappa_0 \wedge \overline{\zeta_0} - \mathscr{T}(k)\ \rho_0 \wedge \kappa_0 \\ d\overline{\zeta_0} = 0. \end{cases}$$

• Initial ambiguity group:

$$g := \begin{pmatrix} \mathsf{c}\overline{\mathsf{c}} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{c} & 0 & 0 & 0 \\ \mathsf{d} & \mathsf{e} & \mathsf{f} & 0 & 0 \\ \overline{\mathsf{b}} & 0 & 0 & \overline{\mathsf{c}} & 0 \\ \overline{\mathsf{d}} & 0 & 0 & \overline{\mathsf{e}} & \overline{\mathsf{f}} \end{pmatrix}.$$

• First normalization:

$$\boxed{\mathsf{f} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\overline{\mathscr{L}_1}(\mathsf{k}).}$$

- **Maurer-Cartan forms at second loop:**

$$\beta^1 := \frac{d\mathsf{c}}{\mathsf{c}},$$
$$\beta^2 := \frac{d\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{c}}{\mathsf{c}^2\overline{\mathsf{c}}},$$
$$\beta^3 := \frac{(-\mathsf{dc}+\mathsf{eb})\,d\mathsf{c}}{\mathsf{c}^3\overline{\mathsf{c}}} - \frac{(-\mathsf{dc}+\mathsf{eb})\,d\overline{\mathsf{c}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{d\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}} - \frac{\mathsf{b}d\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}},$$
$$\beta^4 := -\frac{\mathsf{e}d\mathsf{c}}{\mathsf{c}^2} + \frac{\mathsf{e}d\overline{\mathsf{c}}}{\overline{\mathsf{c}}\mathsf{c}} + \frac{d\mathsf{e}}{\mathsf{c}}.$$

- **Structure:**

$$d\rho = \beta^1\wedge\rho + \overline{\beta^1}\wedge\rho + U^{\rho}_{\rho\kappa}\,\rho\wedge\kappa + U^{\rho}_{\rho\zeta}\,\rho\wedge\zeta + U^{\rho}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa} + U^{\rho}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + i\,\kappa\wedge\overline{\kappa},$$
$$d\kappa = \beta^1\wedge\kappa + \beta^2\wedge\rho + U^{\kappa}_{\rho\kappa}\,\rho\wedge\kappa + U^{\kappa}_{\rho\zeta}\,\rho\wedge\zeta + U^{\kappa}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa}$$
$$+ U^{\kappa}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + U^{\kappa}_{\kappa\zeta}\,\kappa\wedge\zeta + U^{\kappa}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + \zeta\wedge\overline{\kappa},$$
$$d\zeta = \beta^3\wedge\rho + \beta^4\wedge\kappa + \beta^1\wedge\zeta - \overline{\beta^1}\wedge\zeta + U^{\zeta}_{\rho\kappa}\,\rho\wedge\kappa + U^{\zeta}_{\rho\zeta}\,\rho\wedge\zeta + U^{\zeta}_{\rho\overline{\kappa}}\,\rho\wedge\overline{\kappa}$$
$$+ U^{\zeta}_{\rho\overline{\zeta}}\,\rho\wedge\overline{\zeta} + U^{\zeta}_{\kappa\zeta}\,\kappa\wedge\zeta + U^{\zeta}_{\kappa\overline{\kappa}}\,\kappa\wedge\overline{\kappa} + U^{\zeta}_{\kappa\overline{\zeta}}\,\kappa\wedge\overline{\zeta} + U^{\zeta}_{\zeta\overline{\kappa}}\,\zeta\wedge\overline{\kappa} + U^{\zeta}_{\zeta\overline{\zeta}}\,\zeta\wedge\overline{\zeta}.$$

- **Torsion coefficients:**

$$U^{\rho}_{\rho\kappa} = i\,\frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}} + \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1(\mathbf{k})}} + \frac{P}{\mathsf{c}},$$

$$U^{\rho}_{\rho\zeta} = -\frac{\bar{\mathsf{c}}}{\mathsf{c}} \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\rho}_{\rho\bar{\kappa}} = -i\frac{\mathsf{b}}{\mathsf{c}\bar{\mathsf{c}}} + \frac{\bar{\mathsf{e}}\mathsf{c}}{\bar{\mathsf{c}}^2} \frac{\overline{\mathscr{L}_1}\left(\bar{k}\right)}{\mathscr{L}_1\left(\bar{k}\right)} + \frac{\overline{P}}{\bar{\mathsf{c}}},$$

$$U^{\rho}_{\rho\bar{\zeta}} = -\frac{\mathsf{c}}{\bar{\mathsf{c}}} \frac{\overline{\mathscr{L}_1}\left(\bar{k}\right)}{\mathscr{L}_1\left(\bar{k}\right)},$$

$$U^{\kappa}_{\rho\kappa} = -\frac{\mathscr{T}(k)}{\mathsf{c}\bar{\mathsf{c}}} - \frac{\mathsf{e}\bar{\mathsf{b}}}{\mathsf{c}^2\bar{\mathsf{c}}} - \frac{\mathsf{d}}{\mathsf{c}^2} \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} + i\,\frac{\mathsf{b}\bar{\mathsf{b}}}{\mathsf{c}^2\bar{\mathsf{c}}^2} + \frac{\mathsf{b}\mathsf{e}}{\mathsf{c}^3} \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)} + \frac{\mathsf{b}}{\mathsf{c}^2\bar{\mathsf{c}}}\,P,$$

$$U^{\kappa}_{\rho\zeta} = \frac{\bar{\mathsf{b}}}{\mathsf{c}\bar{\mathsf{c}}},$$

$$U^{\kappa}_{\rho\bar{\kappa}} = -\frac{\mathsf{d}}{\mathsf{c}\bar{\mathsf{c}}} + \frac{\mathsf{e}\mathsf{b}}{\mathsf{c}^2\bar{\mathsf{c}}} - i\,\frac{\mathsf{b}^2}{\mathsf{c}^2\bar{\mathsf{c}}^2} + \frac{\mathsf{b}\bar{\mathsf{e}}}{\bar{\mathsf{c}}^3} \frac{\overline{\mathscr{L}_1}\left(\bar{k}\right)}{\mathscr{L}_1\left(\bar{k}\right)} + \frac{\mathsf{b}}{\mathsf{c}\bar{\mathsf{c}}^2}\,\overline{P},$$

$$U^{\kappa}_{\rho\bar{\zeta}} = -\frac{\mathsf{b}}{\bar{\mathsf{c}}^2} \frac{\overline{\mathscr{L}_1}\left(\bar{k}\right)}{\mathscr{L}_1\left(\bar{k}\right)},$$

$$U^{\kappa}_{\kappa\zeta} = -\frac{\bar{\mathsf{c}}}{\mathsf{c}} \frac{\mathscr{L}_1(k)}{\overline{\mathscr{L}_1}(k)},$$

$$U^{\kappa}_{\kappa\overline{\kappa}} = -\frac{\mathsf{e}}{\mathsf{c}} + i\,\frac{\mathsf{b}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$\begin{aligned} U^{\zeta}_{\rho\kappa} = {} & \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{e}\overline{\mathsf{d}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} + \frac{\mathsf{e}\overline{\mathsf{e}}\overline{\mathsf{b}}}{\overline{\mathsf{c}}^3\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\,(k)}{\mathscr{L}_1\left(\overline{k}\right)} + \\ & + \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\frac{\mathscr{T}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \\ & - \frac{\mathsf{e}}{\mathsf{c}^2\overline{\mathsf{c}}}\,\mathscr{T}\,(k) - \frac{\mathsf{e}^2\overline{\mathsf{b}}}{\overline{\mathsf{c}}\mathsf{c}^3} + i\,\frac{\mathsf{d}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}}{\mathsf{c}^2\overline{\mathsf{c}}}\,P, \end{aligned}$$

$$\begin{aligned} U^{\zeta}_{\rho\zeta} = {} & \frac{\overline{\mathsf{d}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{\overline{\mathsf{e}}\overline{\mathsf{b}}}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{\overline{\mathsf{b}}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{b}}{\mathsf{c}^2\overline{\mathsf{c}}}\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} \\ & + \frac{\mathscr{T}}{\mathsf{c}\overline{\mathsf{c}}}\,\frac{\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} + \frac{\mathsf{e}\overline{\mathsf{b}}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{be}}{\mathsf{c}^3}\,\frac{\mathscr{L}_1\,(k)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{d}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\,(k)}{\overline{\mathscr{L}_1}\,(k)}, \end{aligned}$$

$$\begin{aligned} U^{\zeta}_{\rho\overline{\kappa}} = {} & 2\,\frac{\overline{\mathsf{e}}\mathsf{d}}{\overline{\mathsf{c}}^3}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{\mathsf{e}\overline{\mathsf{e}}\mathsf{b}}{\overline{\mathsf{c}}^3\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \\ & - \frac{\mathsf{eb}}{\mathsf{c}^2\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{ed}}{\mathsf{c}^2\overline{\mathsf{c}}} + \frac{\mathsf{e}^2\mathsf{b}}{\overline{\mathsf{c}}\mathsf{c}^3} - i\,\frac{\mathsf{db}}{\mathsf{c}^2\overline{\mathsf{c}}^2} + \frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\overline{P}, \end{aligned}$$

$$U^{\zeta}_{\rho\overline{\zeta}} = -2\,\frac{\mathsf{d}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} + \frac{\mathsf{eb}}{\mathsf{c}\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)},$$

$$U^{\zeta}_{\kappa\zeta} = \frac{1}{\mathsf{c}}\,\frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{e}\overline{\mathsf{c}}}{\mathsf{c}^2}\,\frac{\mathscr{L}_1\,(k)}{\overline{\mathscr{L}_1}\,(k)},$$

$$U^{\zeta}_{\kappa\overline{\kappa}} = \frac{\mathsf{e}\overline{\mathsf{e}}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} + \frac{\mathsf{e}}{\overline{\mathsf{c}}\mathsf{c}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} - \frac{\mathsf{e}^2}{\mathsf{c}^2} + i\,\frac{\mathsf{d}}{\mathsf{c}\overline{\mathsf{c}}},$$

$$U^{\zeta}_{\kappa\overline{\zeta}} = -\frac{\mathsf{e}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)},$$

$$U^{\zeta}_{\zeta\overline{\kappa}} = -\frac{\overline{\mathsf{e}}\mathsf{c}}{\overline{\mathsf{c}}^2}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)} - \frac{1}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\,(k)\right)}{\overline{\mathscr{L}_1}\,(k)} + \frac{\mathsf{e}}{\mathsf{c}},$$

$$U^{\zeta}_{\zeta\overline{\zeta}} = \frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{k}\right)}{\mathscr{L}_1\left(\overline{k}\right)}.$$

- **Normalizable coefficient:**

$$\boxed{\mathsf{b} = -i\,\overline{\mathsf{c}}\mathsf{e} + i\,\frac{\mathsf{c}}{3}\left(\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} - \overline{P}\right).}$$

• Third loop normalization:

$$\boxed{\begin{aligned}\mathsf{d} = & -i\,\frac{1}{2}\,\frac{\mathsf{e}^2\overline{\mathsf{c}}}{\mathsf{c}} + i\,\frac{2}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2} + i\,\frac{1}{18}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\overline{P}}{\overline{\mathscr{L}_1}(k)} \\ & - i\,\frac{1}{9}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{P}^2 + i\,\frac{1}{6}\,.\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - i\,\frac{1}{6}\,\frac{\mathsf{c}}{\overline{\mathsf{c}}}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}.\end{aligned}}$$

Theorem. [Pocchiola] *These two explicit invariants* $\mathbf{J}$ *and* W *whose denominators are related to the nondegeneracy of the Freeman form are fundamental for real analytic hypersurfaces* $M^5 \subset \mathbb{C}^3$ *having Levi form everywhere of rank* $\mathbf{1}$*. Such a hypersurface* M^5 *is biholomorphic to the light cone:*

$$\text{(LC)} \qquad u = \frac{z_1\overline{z}_1 + \frac{1}{2}\,z_1z_1\overline{z}_2 + \frac{1}{2}\,z_2\overline{z}_1\overline{z}_1}{1 - z_2\overline{z}_2} \underset{\text{locally}}{\cong} (\mathsf{Re}\,z_1')^2 - (\mathsf{Re}\,z_2')^2 - (\mathsf{Re}\,z_3')^2,$$

which possesses a group of local biholomorphic automorphisms of dimension $\mathbf{10}$*:*

$$\mathsf{Aut}_{CR}(\mathsf{LC}) \cong \mathsf{Sp}(4,\mathbb{R}),$$

if and only if:

$$0 \equiv W \equiv J.$$

Furthermore, if $W \not\equiv 0$ or if $J \not\equiv 0$, a canonical absolute parallelism exists on M, and in this case, the automorphism group of M always satisfies:

$$\mathsf{dim}\,\mathsf{Aut}_{CR}(M) \leqslant \mathbf{5}.$$

- **First fundamental invariant:**

$$\begin{aligned}\mathbf{W} := \frac{\mathscr{L}_1\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{2}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)} + \frac{1}{3}\frac{\mathscr{K}\left(\overline{P}\right)}{\overline{\mathscr{L}_1}(k)} + \\ + \frac{1}{3}\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\mathscr{K}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)^3} - \frac{1}{3}\frac{\mathscr{K}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \frac{P}{3}.\end{aligned}$$

• Second fundamental invariant:

$$
\begin{aligned}
\overline{\mathbf{J}} = {} & \frac{5}{18}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^2}{\overline{\mathscr{L}_1}(k)^2}\,\overline{P} + \frac{1}{3}\,\overline{P}\,\overline{\mathscr{L}_1}\left(\overline{P}\right) - \frac{1}{9}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P}^2 + \\
& + \frac{20}{27}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)^3}{\overline{\mathscr{L}_1}(k)^3} - \frac{5}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)^2} + \\
& + \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\,\overline{\mathscr{L}_1}(\overline{P})}{\overline{\mathscr{L}_1}(k)} - \frac{1}{6}\,\frac{\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}(k)\right)\right)}{\overline{\mathscr{L}_1}(k)}\,\overline{P} - \\
& - \frac{2}{27}\,\overline{P}^3 - \frac{1}{6}\,\overline{\mathscr{L}_1}\left(\overline{\mathscr{L}_1}\left(\overline{P}\right)\right).
\end{aligned}
$$

Deformations of the Beloshapka Cubic

[Merker-Sabzevari 2016]

• **Coordinates on $\mathbb{C}^4$:**

$$\big(z, w_1, w_2, w_3\big) \in \mathbb{C}^4.$$

• **Model cubic of CR dimension $n = 1$ in $\mathbb{C}^4$:**

[Beloshapka 2000]

$$\begin{bmatrix} w_1 - \underline{w}_1 = 2iz\underline{z}, \\ w_2 - \underline{w}_2 = 2iz\underline{z}(z + \underline{z}), \\ w_3 - \underline{w}_3 = 2z\underline{z}(z - \underline{z}). \end{bmatrix}$$

Proposition. *The Lie algebra:*

$$\mathfrak{aut}_{CR}(M) = 2\,\mathrm{Re}\,\mathfrak{hol}(M)$$

of infinitesimal CR automorphisms of the model cubic is **7***-dimensional, generated by the real parts of the seven holomorphic vector fields* $\mathbb{R}$*-linearly independent:*

$$
\begin{aligned}
T &:= \partial_{w_1}, \\
S_1 &:= \partial_{w_2}, \\
S_2 &:= \partial_{w_3}, \\
L_1 &:= \partial_z + (2iz)\,\partial_{w_1} + (2iz^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}, \\
L_2 &:= i\,\partial_z + (2z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2iz^2 - 4w_1)\,\partial_{w_3}, \\
D &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}, \\
R &:= iz\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned}
$$

- **Commutation table:**

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	$*$	0	0	0	0	$3S_1$	S_2
T	$*$	$*$	0	$4S_2$	$4S_1$	$2T$	0
L_2	$*$	$*$	$*$	0	$-4T$	L_2	$-L_1$
L_1	$*$	$*$	$*$	$*$	0	L_1	L_2
D	$*$	$*$	$*$	$*$	$*$	0	0
R	$*$	$*$	$*$	$*$	$*$	$*$	$0.$

- **Set:**

$$\mathfrak{g} := \mathfrak{aut}_{CR}(M),$$

- **Decompose:**

$$\begin{aligned}
\mathfrak{g}_{-3} &:= \mathsf{Span}_{\mathbb{R}}\big\langle S_1,\, S_2\big\rangle,\\
\mathfrak{g}_{-2} &:= \mathsf{Span}_{\mathbb{R}}\big\langle T\big\rangle,\\
\mathfrak{g}_{-1} &:= \mathsf{Span}_{\mathbb{R}}\big\langle L_1, L_2\big\rangle,\\
\mathfrak{g}_{0} &:= \mathsf{Span}_{\mathbb{R}}\big\langle D, R\big\rangle.
\end{aligned}$$

- **Natural graduation:**

$$\mathfrak{g} = \underbrace{\mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}}_{\substack{\cong\, \mathfrak{n}_5^4 \text{ classification} \\ \text{of Goze-Remm}}} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

- **Coordinates:**

$$\big(zx + iy,\ u_1 + iv_1,\ u_2 + iv_2,\ u_3 + iv_3\big) \ \in\ \mathbb{C}^4.$$

- 5**-dimensional graph:**

$$\begin{aligned} v_1 &= \varphi_1\big(x, y, u_1, u_2, u_3\big), \\ v_2 &= \varphi_2\big(x, y, u_1, u_2, u_3\big), \\ v_3 &= \varphi_3\big(x, y, u_1, u_2, u_3\big). \end{aligned}$$

- **Generator of $T^{0,1}M$:**

$$\overline{\mathscr{L}} = \frac{\partial}{\partial \overline{z}} + \mathsf{A}_1 \frac{\partial}{\partial \overline{w}_1} + \mathsf{A}_2 \frac{\partial}{\partial \overline{w}_2} + \mathsf{A}_3 \frac{\partial}{\partial \overline{w}_3}.$$

- **Expression of its coefficients:**

$$\mathsf{A}_1 = \frac{\Lambda_1^1}{\Delta} + i\frac{\Lambda_2^1}{\Delta},$$
$$\mathsf{A}_2 = \frac{\Lambda_1^2}{\Delta} + i\frac{\Lambda_2^2}{\Delta},$$
$$\mathsf{A}_3 = \frac{\Lambda_1^3}{\Delta} + i\frac{\Lambda_2^3}{\Delta}.$$

- **Denominator:**

$$\Delta = \sigma^2 + \tau^2,$$

- **with:**

$$\sigma = \varphi_{3u_3} + \varphi_{1u_1} + \varphi_{2u_2} - \varphi_{1u_2}\varphi_{3u_1}\varphi_{2u_3} - \varphi_{1u_3}\varphi_{2u_1}\varphi_{3u_2} + \varphi_{1u_2}\varphi_{2u_1}\varphi_{3u_3} -$$
$$- \varphi_{1u_1}\varphi_{2u_2}\varphi_{3u_3} + \varphi_{1u_1}\varphi_{2u_3}\varphi_{3u_2} + \varphi_{1u_3}\varphi_{3u_1}\varphi_{2u_2},$$
$$\tau = -1 + \varphi_{1u_1}\varphi_{2u_2} - \varphi_{2u_3}\varphi_{3u_2} - \varphi_{1u_3}\varphi_{3u_1} + \varphi_{2u_2}\varphi_{3u_3} - \varphi_{1u_2}\varphi_{2u_1} + \varphi_{1u_1}\varphi_{3u_3}.$$

- **Numerators:**

$$
\begin{aligned}
\Lambda_1^1 = \Big(&- \varphi_{3u_3}\varphi_{2x}\varphi_{1u_2} - \varphi_{1u_3}\varphi_{3y} + \varphi_{2u_2}\varphi_{1x}\varphi_{3u_3} + \varphi_{3u_3}\varphi_{1y} - \varphi_{1x} - \varphi_{2y}\varphi_{1u_2} + \\
&+ \varphi_{2u_3}\varphi_{3x}\varphi_{1u_2} + \varphi_{2u_2}\varphi_{1y} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3x} + \varphi_{2x}\varphi_{1u_3}\varphi_{3u_2}\Big)\sigma + \\
&+ \Big(\varphi_{1u_3}\varphi_{3x} - \varphi_{1y} + \varphi_{2x}\varphi_{1u_2} + \varphi_{2u_3}\varphi_{1u_2}\varphi_{3y} - \varphi_{2u_2}\varphi_{1x} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1y} - \\
&- \varphi_{3u_3}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3y} - \varphi_{3u_3}\varphi_{1u_2}\varphi_{2y} + \varphi_{1u_3}\varphi_{3u_2}\varphi_{2y} + \varphi_{2u_2}\varphi_{3u_3}\varphi_{1y}\Big)\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2^1 = \Big(&\varphi_{1u_3}\varphi_{3x} - \varphi_{1y} + \varphi_{2x}\varphi_{1u_2} + \varphi_{2u_3}\varphi_{1u_2}\varphi_{3y} - \varphi_{2u_2}\varphi_{1x} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1y} - \\
&- \varphi_{3u_3}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3y} - \varphi_{3u_3}\varphi_{1u_2}\varphi_{2y} + \varphi_{1u_3}\varphi_{3u_2}\varphi_{2y} + \varphi_{2u_2}\varphi_{3u_3}\varphi_{1y}\Big)\sigma - \\
&- \Big(- \varphi_{3u_3}\varphi_{2x}\varphi_{1u_2} - \varphi_{1u_3}\varphi_{3y} + \varphi_{2u_2}\varphi_{1x}\varphi_{3u_3} + \varphi_{3u_3}\varphi_{1y} - \varphi_{1x} - \varphi_{2y}\varphi_{1u_2} + \\
&+ \varphi_{2u_3}\varphi_{3x}\varphi_{1u_2} + \varphi_{2u_2}\varphi_{1y} - \varphi_{2u_3}\varphi_{3u_2}\varphi_{1x} - \varphi_{2u_2}\varphi_{1u_3}\varphi_{3x} + \varphi_{2x}\varphi_{1u_3}\varphi_{3u_2}\Big)\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_1^2 = \Big(&- \varphi_{2x} + \varphi_{3u_3}\varphi_{2y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3x} - \varphi_{2u_3}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2x} - \varphi_{2u_1}\varphi_{1y} - \\
&- \varphi_{2u_1}\varphi_{3u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{2y} - \varphi_{1u_1}\varphi_{2u_3}\varphi_{3x} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2x}\Big)\sigma + \\
&+ \Big(- \varphi_{1u_1}\varphi_{2u_3}\varphi_{3y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2y} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1y} + \varphi_{2u_3}\varphi_{3x} - \\
&- \varphi_{2u_1}\varphi_{3u_3}\varphi_{1y} - \varphi_{3u_3}\varphi_{2x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2y} + \varphi_{2u_1}\varphi_{1x} - \varphi_{1u_1}\varphi_{2x} - \varphi_{2y}\Big)\tau,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_2^2 = \Big(&- \varphi_{1u_1}\varphi_{2u_3}\varphi_{3y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2y} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1y} + \varphi_{2u_3}\varphi_{3x} - \\
&- \varphi_{2u_1}\varphi_{3u_3}\varphi_{1y} - \varphi_{3u_3}\varphi_{2x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2y} + \varphi_{2u_1}\varphi_{1x} - \varphi_{1u_1}\varphi_{2x} - \varphi_{2y}\Big)\sigma - \\
&- \Big(- \varphi_{2x} + \varphi_{3u_3}\varphi_{2y} + \varphi_{1u_3}\varphi_{2u_1}\varphi_{3x} - \varphi_{2u_3}\varphi_{3y} - \varphi_{1u_3}\varphi_{3u_1}\varphi_{2x} - \varphi_{2u_1}\varphi_{1y} - \\
&- \varphi_{2u_1}\varphi_{3u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{2y} - \varphi_{1u_1}\varphi_{2u_3}\varphi_{3x} + \varphi_{3u_1}\varphi_{2u_3}\varphi_{1x} + \varphi_{1u_1}\varphi_{3u_3}\varphi_{2x}\Big)\tau,
\end{aligned}
$$

$$\begin{aligned}
\Lambda_1^3 = \Big(& -\varphi_{2u_1}\varphi_{1u_2}\varphi_{3x} - \varphi_{3u_1}\varphi_{1y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1x} + \varphi_{1u_1}\varphi_{3y} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1x} - \\
& - \varphi_{3u_2}\varphi_{2y} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2x} + \varphi_{2u_2}\varphi_{3y} - \varphi_{3u_2}\varphi_{1u_1}\varphi_{2x} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3x} - \varphi_{3x}\Big)\sigma + \\
& + \Big(-\varphi_{3u_2}\varphi_{1u_1}\varphi_{2y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1y} + \varphi_{3u_1}\varphi_{1x} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2y} - \varphi_{2u_1}\varphi_{1u_2}\varphi_{3y} + \\
& + \varphi_{3u_2}\varphi_{2x} - \varphi_{1u_1}\varphi_{3x} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1y} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3y} - \varphi_{3y} - \varphi_{2u_2}\varphi_{3x}\Big)\tau, \\
\Lambda_2^3 = \Big(& -\varphi_{3u_2}\varphi_{1u_1}\varphi_{2y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1y} + \varphi_{3u_1}\varphi_{1x} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2y} - \varphi_{2u_1}\varphi_{1u_2}\varphi_{3y} + \\
& + \varphi_{3u_2}\varphi_{2x} - \varphi_{1u_1}\varphi_{3x} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1y} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3y} - \varphi_{3y} - \varphi_{2u_2}\varphi_{3x}\Big)\sigma - \\
& - \Big(-\varphi_{2u_1}\varphi_{1u_2}\varphi_{3x} - \varphi_{3u_1}\varphi_{1y} + \varphi_{2u_1}\varphi_{3u_2}\varphi_{1x} + \varphi_{1u_1}\varphi_{3y} - \varphi_{3u_1}\varphi_{2u_2}\varphi_{1x} - \\
& - \varphi_{3u_2}\varphi_{2y} + \varphi_{3u_1}\varphi_{1u_2}\varphi_{2x} + \varphi_{2u_2}\varphi_{3y} - \varphi_{3u_2}\varphi_{1u_1}\varphi_{2x} + \varphi_{1u_1}\varphi_{2u_2}\varphi_{3x} - \varphi_{3x}\Big)\tau.
\end{aligned}$$

- **Third independent field:**

$$\mathscr{T} := i\left[\mathscr{L}, \overline{\mathscr{L}}\right].$$

- **Direct:**

$$\boxed{\mathscr{T} = \frac{\Upsilon_1}{\Delta^3}\frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^3}\frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^3}\frac{\partial}{\partial u_3}.}$$

- **Expressions of the numerators:**

$$\begin{aligned}
\Upsilon_1 = -\big(&\Delta^2\Lambda^1_{2x} - \Delta\Delta_x\Lambda^1_2 - \Delta^2\Lambda^1_{1y} + \Delta\Delta_y\Lambda^1_1 + \Delta\Lambda^1_1\Lambda^1_{2u_1} - \Delta\Lambda^1_2\Lambda^1_{1u_1} - \Delta\Lambda^2_2\Lambda^1_{1u_2} + \\
&+ \Delta_{u_2}\Lambda^1_1\Lambda^2_2 - \Delta\Lambda^3_2\Lambda^1_{1u_3} + \Delta_{u_3}\Lambda^3_2\Lambda^1_1 + \Delta\Lambda^2_1\Lambda^1_{2u_2} - \Delta_{u_2}\Lambda^2_1\Lambda^1_2 + \Delta\Lambda^3_1\Lambda^1_{2u_3} - \Delta_{u_3}\Lambda^3_1\Lambda^1_2\big), \\
\Upsilon_2 = -\big(&\Delta^2\Lambda^2_{2x} - \Delta\Delta_x\Lambda^2_2 + \Delta\Lambda^1_1\Lambda^2_{2u_1} - \Delta_{u_1}\Lambda^1_1\Lambda^2_2 - \Delta^2\Lambda^2_{1y} + \Delta\Delta_y\Lambda^2_1 - \Delta\Lambda^1_2\Lambda^2_{1u_1} + \\
&+ \Delta_{u_1}\Lambda^1_2\Lambda^2_1 + \Delta\Lambda^2_1\Lambda^2_{2u_2} - \Delta\Lambda^2_2\Lambda^2_{1u_2} + \Delta\Lambda^3_1\Lambda^2_{2u_3} - \Delta_{u_3}\Lambda^3_1\Lambda^2_2 - \Delta\Lambda^3_2\Lambda^2_{1u_3} + \Delta_{u_3}\Lambda^3_2\Lambda^2_1\big), \\
\Upsilon_3 = -\big(&\Delta^2\Lambda^3_{2x} - \Delta\Delta_x\Lambda^3_2 + \Delta\Lambda^1_1\Lambda^3_{2u_1} - \Delta_{u_1}\Lambda^1_1\Lambda^3_2 - \Delta^2\Lambda^3_{1y} + \Delta\Delta_y\Lambda^3_1 - \Delta\Lambda^1_2\Lambda^3_{1u_1} + \\
&+ \Delta_{u_1}\Lambda^1_2\Lambda^3_1 - \Delta\Lambda^2_2\Lambda^3_{1u_2} + \Delta_{u_2}\Lambda^2_2\Lambda^3_1 + \Delta\Lambda^3_1\Lambda^3_{2u_3} - \Delta\Lambda^3_2\Lambda^3_{1u_3} + \Delta\Lambda^2_1\Lambda^3_{2u_2} - \Delta_{u_2}\Lambda^2_1\Lambda^3_2\big).
\end{aligned}$$

- **Last 2 fields completing a frame:**

$$\begin{aligned}
\mathscr{S} &:= [\mathscr{L}, \mathscr{T}], \\
\overline{\mathscr{S}} &:= [\overline{\mathscr{L}}, \mathscr{T}].
\end{aligned}$$

- **Notational contraction:**

$$\mathscr{S} = \frac{\Gamma^1_1 - i\Gamma^1_2}{\Delta^5}\frac{\partial}{\partial u_1} + \frac{\Gamma^2_1 - i\Gamma^2_2}{\Delta^5}\frac{\partial}{\partial u_2} + \frac{\Gamma^3_1 - i\Gamma^3_2}{\Delta^5}\frac{\partial}{\partial u_3}.$$

- **Partial expansions:**

$$
\begin{aligned}
\Gamma_i^1 &= -2\big(\tfrac{1}{4}\Delta^2\Upsilon_{1x_i} - 3\Delta\Delta_{x_i}\Upsilon_1 + \Delta\Lambda_i^1\Upsilon_{1u_1} - 2\Delta_{u_1}\Lambda_i^1\Upsilon_1 - \Delta\Lambda_{iu_1}^1\Upsilon_1 - \Delta\Lambda_{iu_2}^1\Upsilon_2 + \\
&\quad + \Delta_{u_2}\Lambda_i^1\Upsilon_2 - \Delta\Lambda_{iu_3}^1\Upsilon_3 + \Delta_{u_3}\Lambda_i^1\Upsilon_3 + \Delta\Lambda_i^2\Upsilon_{1u_2} - 3\Delta_{u_2}\Lambda_i^2\Upsilon_1 + \Delta\Lambda_i^3\Upsilon_{1u_3} - 3\Delta_{u_3}\Lambda_i^3\Upsilon_1\big),\\
\Gamma_i^2 &= -2\big(\Delta^2\Upsilon_{2x_i} - 3\Delta\Delta_{x_i}\Upsilon_2 + \Delta\Lambda_i^1\Upsilon_{2u_1} - 3\Delta_{u_1}\Lambda_i^1\Upsilon_2 - \Delta\Lambda_{iu_1}^2\Upsilon_1 + \Delta_{u_1}\Lambda_i^2\Upsilon_1 + \\
&\quad + \Delta\Lambda_i^2\Upsilon_{2u_2} - 2\Delta_{u_2}\Lambda_i^2\Upsilon_2 - \Delta\Lambda_{iu_2}^2\Upsilon_2 - \Delta\Lambda_{iu_3}^2\Upsilon_3 + \Delta_{u_3}\Lambda_i^2\Upsilon_3 + \Delta\Lambda_i^3\Upsilon_{2u_3} - 3\Delta_{u_3}\Lambda_i^3\Upsilon_2\big),\\
\Gamma_i^3 &= -2\big(\Delta^2\Upsilon_{3x_i} - 3\Delta\Delta_{x_i}\Upsilon_3 + \Delta\Lambda_i^1\Upsilon_{3u_1} - 3\Delta_{u_1}\Lambda_i^1\Upsilon_3 + \Delta\Lambda_i^2\Upsilon_{3u_2} - 3\Delta_{u_2}\Lambda_i^2\Upsilon_3 - \\
&\quad - \Delta\Lambda_{iu_1}^3\Upsilon_1 + \Delta_{u_1}\Lambda_i^3\Upsilon_1 - \Delta\Lambda_{iu_2}^3\Upsilon_2 + \Delta_{u_2}\Lambda_i^3\Upsilon_2 + \Delta\Lambda_i^3\Upsilon_{3u_3} - 2\Delta_{u_3}\Lambda_i^3\Upsilon_3 - \Delta\Lambda_{iu_3}^3\Upsilon_3\big).
\end{aligned}
$$

Even at the starting point, the data explode!

- **Five fields making up a frame:**

$$
\Big\{\overline{\mathscr{S}},\ \mathscr{S},\ \mathscr{T},\ \overline{\mathscr{L}},\ \mathscr{L}\Big\},
$$

where:

$$
\begin{aligned}
\mathscr{T} &:= i\,\big[\mathscr{L}, \overline{\mathscr{L}}\big],\\
\mathscr{S} &:= \big[\mathscr{L},\ \mathscr{T}\big],\\
\overline{\mathscr{S}} &:= \big[\overline{\mathscr{L}},\ \mathscr{T}\big].
\end{aligned}
$$

- **Length 4 brackets:**

$$\boxed{\begin{aligned} \big[\mathscr{L},\,\mathscr{S}\big] &= P\,\mathscr{T} + Q\,\mathscr{S} + R\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\,\mathscr{S}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}, \\ \big[\mathscr{L},\,\overline{\mathscr{S}}\big] &= A\,\mathscr{T} + B\,\mathscr{S} + \overline{B}\,\overline{\mathscr{S}}, \\ \big[\overline{\mathscr{L}},\,\overline{\mathscr{S}}\big] &= \overline{P}\,\mathscr{T} + \overline{R}\,\mathscr{S} + \overline{Q}\,\overline{\mathscr{S}}. \end{aligned}}$$

- **Length 5 brackets:**

$$\begin{aligned} \big[\mathscr{T},\,\mathscr{S}\big] &= \Big[i[\mathscr{L},\overline{\mathscr{L}}],\ \big[\mathscr{L},\,i[\mathscr{L},\overline{\mathscr{L}}]\big]\Big], \\ \big[\mathscr{T},\,\overline{\mathscr{S}}\big] &= \Big[i[\mathscr{L},\overline{\mathscr{L}}],\ \big[\overline{\mathscr{L}},\,i[\mathscr{L},\overline{\mathscr{L}}]\big]\Big]. \end{aligned}$$

Lemma. *The coefficients of the two brackets:*

$$\begin{aligned} \big[\mathscr{T},\,\mathscr{S}\big] &= E\,\mathscr{T} + F\,\mathscr{S} + G\,\overline{\mathscr{S}}, \\ \big[\mathscr{T},\,\overline{\mathscr{S}}\big] &= \overline{E}\,\mathscr{T} + \overline{G}\,\mathscr{S} + \overline{F}\,\overline{\mathscr{S}}, \end{aligned}$$

are the functions:

$$\begin{aligned} E &= -i\,\overline{\mathscr{L}}(P) - i\,A\,Q - i\,\overline{P}\,R + i\,\mathscr{L}(A) + i\,B\,P + i\,A\,\overline{B}, \\ F &= -i\,\overline{\mathscr{L}}(Q) - i\,R\,\overline{R} + i\,A + i\,\mathscr{L}(B) + i\,B\,\overline{B}, \\ G &= -i\,P - i\,\overline{B}\,Q - i\,R\,\overline{Q} - i\,\overline{\mathscr{L}}(R) + i\,B\,R + i\,\overline{B}\overline{B} + i\,\mathscr{L}(\overline{B}). \end{aligned}$$

- **Length 6 brackets:**

$$\big[\mathscr{S},\,\overline{\mathscr{S}}\big]=\Big[\big[\mathscr{L},\,i[\mathscr{L},\overline{\mathscr{L}}]\big],\ \big[\overline{\mathscr{L}},\,i[\mathscr{L},\overline{\mathscr{L}}]\big]\Big].$$

Lemma. *The coefficients of the last bracket:*

$$\big[\mathscr{S},\,\overline{\mathscr{S}}\big]=i\,J\,\mathscr{T}+K\,\mathscr{S}-\overline{K}\,\overline{\mathscr{S}},$$

are the functions:

$$\begin{aligned}
-\,2\,J=&-\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(P)\big)+\overline{\mathscr{L}}\big(\mathscr{L}(A)\big)+\mathscr{L}\big(\overline{\mathscr{L}}(A)\big)-\mathscr{L}\big(\mathscr{L}(\overline{P})\big)\\
&-\,Q\,\overline{\mathscr{L}}(A)-2\,A\,\overline{\mathscr{L}}(Q)-R\,\overline{\mathscr{L}}(\overline{P})-2\,\overline{P}\,\overline{\mathscr{L}}(R)-2\,AR\overline{R}-2\,P\overline{P}-\overline{BP}Q-\overline{PQ}R-\\
&-\overline{R}\mathscr{L}(P)-2\,P\,\mathscr{L}(\overline{R})-\overline{Q}\,\mathscr{L}(A)-2\,A\mathscr{L}(\overline{Q})-PQ\overline{R}-BP\overline{Q}+\\
&+2\,P\overline{\mathscr{L}}(B)+B\overline{\mathscr{L}}(P)+2\,A\overline{\mathscr{L}}(\overline{B})+\overline{B}\,\overline{\mathscr{L}}(A)+2\,A\,\mathscr{L}(B)+2\,AA+2\,AB\overline{B}+2\,\overline{P}\,\mathscr{L}(\overline{B})+\\
&+B\overline{P}R+\overline{BBP}+B\,\mathscr{L}(A)+\overline{B}\,\mathscr{L}(\overline{P})+BBP+\overline{B}P\overline{R},
\end{aligned}$$

$$\begin{aligned}
2i\,K=&-\overline{\mathscr{L}}\big(\overline{\mathscr{L}}(Q)\big)+\overline{\mathscr{L}}\big(\mathscr{L}(B)\big)+\mathscr{L}\big(\overline{\mathscr{L}}(B)\big)-\mathscr{L}\big(\mathscr{L}(\overline{R})\big)-\\
&-2\,\overline{R}\,\overline{\mathscr{L}}(R)-R\,\overline{\mathscr{L}}(\overline{R})-B\,\overline{\mathscr{L}}(Q)-BR\overline{R}-2\,P\overline{R}-\overline{Q}R\overline{R}-2\,\mathscr{L}(\overline{P})-\overline{R}\,\mathscr{L}(Q)-\\
&-\,2\,Q\,\mathscr{L}(\overline{R})-\overline{Q}\,\mathscr{L}(B)-2\,B\,\mathscr{L}(\overline{Q})-A\overline{Q}-\overline{P}Q-QQ\overline{R}-BQ\overline{Q}+\\
&+2\,\overline{\mathscr{L}}(A)+\overline{B}\,\overline{\mathscr{L}}(B)+2\,B\,\overline{\mathscr{L}}(\overline{B})+3\,B\,\mathscr{L}(B)+3\,AB+BBQ+2\,BB\overline{B}+2\,\overline{R}\,\mathscr{L}(\overline{B})+\\
&+\overline{BBR}+\overline{B}\,\mathscr{L}(\overline{R})+\overline{BP}+Q\,\overline{\mathscr{L}}(B).
\end{aligned}$$

• Relations from Jacobi:

$$
\begin{aligned}
0 &\overset{1}{=} \Big[\mathscr{L},\, \Big[\mathscr{L},\, [\mathscr{L},\, [\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] - 2\Big[\mathscr{L},\, \Big[\overline{\mathscr{L}},\, [\mathscr{L},\, [\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] + \Big[\overline{\mathscr{L}},\, \Big[\mathscr{L},\, [\mathscr{L},\, [\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big], \\
0 &\overset{2}{=} \Big[\overline{\mathscr{L}},\, \Big[\overline{\mathscr{L}},\, [\overline{\mathscr{L}},\, [\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] - 2\Big[\overline{\mathscr{L}},\, \Big[\mathscr{L},\, [\overline{\mathscr{L}},\, [\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] + \Big[\mathscr{L},\, \Big[\overline{\mathscr{L}},\, [\overline{\mathscr{L}},\, [\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big], \\
0 &\overset{3}{=} \Big[\mathscr{L},\, \Big[\mathscr{L},\, [\overline{\mathscr{L}},\, [\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] - 3\Big[\mathscr{L},\, \Big[\overline{\mathscr{L}},\, [\mathscr{L},\, [\overline{\mathscr{L}},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] + 3\Big[\overline{\mathscr{L}},\, \Big[\mathscr{L},\, [\overline{\mathscr{L}},\, [\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big] - \\
&\qquad - \Big[\overline{\mathscr{L}},\, \Big[\overline{\mathscr{L}},\, [\mathscr{L},\, [\mathscr{L},[\mathscr{L},\overline{\mathscr{L}}]]]\Big]\Big].
\end{aligned}
$$

• Frame and dual coframe:

$$
\big\{du_3, du_2, du_1,\, dz,\, d\overline{z}\big\} \quad \Big\{\frac{\partial}{\partial u_3},\, \frac{\partial}{\partial u_2},\, \frac{\partial}{\partial u_1},\, \frac{\partial}{\partial z},\, \frac{\partial}{\partial \overline{z}}\Big\}.
$$

• Introduce:

$$
\big\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\big\} \quad \text{dual of} \quad \big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\}.
$$

• By definition:

$$
\begin{array}{lllll}
\overline{\sigma_0}(\overline{\mathscr{S}}) = 1 & \overline{\sigma_0}(\mathscr{S}) = 0 & \overline{\sigma_0}(\mathscr{T}) = 0 & \overline{\sigma_0}(\overline{\mathscr{L}}) = 0 & \overline{\sigma_0}\big(\mathscr{L}\big) = 0, \\
\sigma_0(\overline{\mathscr{S}}) = 0 & \sigma_0(\mathscr{S}) = 1 & \sigma_0(\mathscr{T}) = 0 & \sigma_0(\overline{\mathscr{L}}) = 0 & \sigma_0\big(\mathscr{L}\big) = 0, \\
\rho_0(\overline{\mathscr{S}}) = 0 & \rho_0(\mathscr{S}) = 0 & \rho_0(\mathscr{T}) = 1 & \rho_0(\overline{\mathscr{L}}) = 0 & \rho_0(\mathscr{L}) = 0, \\
\overline{\zeta_0}(\overline{\mathscr{S}}) = 0 & \overline{\zeta_0}(\mathscr{S}) = 0 & \overline{\zeta_0}(\mathscr{T}) = 0 & \overline{\zeta_0}(\overline{\mathscr{L}}) = 1 & \overline{\zeta_0}\big(\mathscr{L}\big) = 0, \\
\zeta_0(\overline{\mathscr{S}}) = 0 & \zeta_0(\mathscr{S}) = 0 & \zeta_0(\mathscr{T}) = 0 & \zeta_0(\overline{\mathscr{L}}) = 0 & \zeta_0\big(\mathscr{L}\big) = 1.
\end{array}
$$

• Observation:

$$
\zeta_0 = dz \qquad\qquad \text{et} \qquad\qquad \overline{\zeta_0} = d\overline{z}.
$$

- **Lie structure:**

$$\big[\mathscr{L}_{i_1},\,\mathscr{L}_{i_2}\big]=\sum_{k=1}^{n}a_{i_1,i_2}^{k}\,\mathscr{L}_k \qquad\qquad (1\leqslant i_1<i_2\leqslant n).$$

- **Duale Darboux-Cartan structure:**

$$d\omega^k=-\sum_{1\leqslant i_1<i_2\leqslant n}a_{i_1,i_2}^{k}\,\omega^{i_1}\wedge\omega^{i_2} \qquad\qquad (k=1\cdots n).$$

- **Array:**

$$
\begin{array}{rcccccccccccc}
 & & \overline{\mathscr{S}} & & \mathscr{S} & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & & \\
 & & \boxed{d\overline{\sigma_0}} & & \boxed{d\sigma_0} & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & & \\
\big[\overline{\mathscr{S}},\,\mathscr{S}\big] & = & \overline{K}\cdot\overline{\mathscr{S}} & + & -K\cdot\mathscr{S} & + & -i\,J\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\sigma_0} \\
\big[\overline{\mathscr{S}},\,\mathscr{T}\big] & = & -\overline{F}\cdot\overline{\mathscr{S}} & + & -\overline{G}\cdot\mathscr{S} & + & -\overline{E}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\rho_0} \\
\big[\overline{\mathscr{S}},\,\overline{\mathscr{L}}\big] & = & -\overline{Q}\cdot\overline{\mathscr{S}} & + & -\overline{R}\cdot\mathscr{S} & + & -\overline{P}\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\overline{\zeta_0}} \\
\big[\overline{\mathscr{S}},\,\mathscr{L}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\sigma_0}\wedge\zeta_0} \\
\big[\mathscr{S},\,\mathscr{T}\big] & = & -G\cdot\overline{\mathscr{S}} & + & -F\cdot\mathscr{S} & + & -E\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\rho_0} \\
\big[\mathscr{S},\,\overline{\mathscr{L}}\big] & = & -\overline{B}\cdot\overline{\mathscr{S}} & + & -B\cdot\mathscr{S} & + & -A\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{S},\,\mathscr{L}\big] & = & -R\cdot\overline{\mathscr{S}} & + & -Q\cdot\mathscr{S} & + & -P\cdot\mathscr{T} & + & 0 & + & 0 & & \boxed{\sigma_0\wedge\zeta_0} \\
\big[\mathscr{T},\,\overline{\mathscr{L}}\big] & = & -\overline{\mathscr{S}} & + & 0 & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\overline{\zeta_0}} \\
\big[\mathscr{T},\,\mathscr{L}\big] & = & 0 & + & -\mathscr{S} & + & 0 & + & 0 & + & 0 & & \boxed{\rho_0\wedge\zeta_0} \\
\big[\overline{\mathscr{L}},\,\mathscr{L}\big] & = & 0 & + & 0 & + & i\,\mathscr{T} & + & 0 & + & 0 & & \boxed{\overline{\zeta_0}\wedge\zeta_0}
\end{array}
$$

- **Read vertically and change signs:**

$$\boxed{\begin{aligned} d\overline{\sigma}_0 &= -\,\overline{K}\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{F}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{Q}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+\overline{B}\cdot\overline{\sigma}_0\wedge\zeta_0+\\ &\qquad +G\cdot\sigma_0\wedge\rho_0+\overline{B}\cdot\sigma_0\wedge\overline{\zeta}_0+R\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\overline{\zeta}_0,\\ d\sigma_0 &= K\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{G}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{R}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+B\cdot\overline{\sigma}_0\wedge\zeta_0+\\ &\qquad +F\cdot\sigma_0\wedge\rho_0+B\cdot\sigma_0\wedge\overline{\zeta}_0+Q\cdot\sigma_0\wedge\zeta_0+\rho_0\wedge\zeta_0,\\ d\rho_0 &= i\,J\cdot\overline{\sigma}_0\wedge\sigma_0+\overline{E}\cdot\overline{\sigma}_0\wedge\rho_0+\overline{P}\cdot\overline{\sigma}_0\wedge\overline{\zeta}_0+A\cdot\overline{\sigma}_0\wedge\zeta_0+\\ &\qquad +E\cdot\sigma_0\wedge\rho_0+A\cdot\sigma_0\wedge\overline{\zeta}_0+P\cdot\sigma_0\wedge\zeta_0-i\,\overline{\zeta}_0\wedge\zeta_0,\\ d\overline{\zeta}_0 &= 0,\\ d\zeta_0 &= 0. \end{aligned}}$$

- **Ambiguity group of G-structure:**

$$G := \left\{ g = \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0\\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0\\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a} \end{pmatrix}, \qquad \mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d},\mathsf{e}\in\mathbb{C} \right\}.$$

- **Lifted coframe:**

$$\begin{pmatrix}\overline{\sigma}\\ \sigma\\ \rho\\ \overline{\zeta}\\ \zeta\end{pmatrix} := \underbrace{\begin{pmatrix}\mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0\\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0\\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0\\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0\\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a}\end{pmatrix}}_{=:g} \begin{pmatrix}\overline{\sigma_0}\\ \sigma_0\\ \rho_0\\ \overline{\zeta_0}\\ \zeta_0\end{pmatrix}.$$

- **Structure equations:**

$$\begin{aligned} d\sigma = \big(2\,\alpha_1 + \overline{\alpha}_1\big) \wedge \sigma + \\ + U_1\,\sigma \wedge \overline{\sigma} + U_2\,\sigma \wedge \rho + U_3\,\sigma \wedge \zeta + U_4\,\sigma \wedge \overline{\zeta} + \\ + U_5\,\overline{\sigma} \wedge \rho + U_6\,\overline{\sigma} \wedge \zeta + U_7\,\overline{\sigma} \wedge \overline{\zeta} + \\ + \rho \wedge \zeta, \end{aligned}$$

$$
\begin{aligned}
d\rho = {} & \alpha_2 \wedge \sigma + \overline{\alpha}_2 \wedge \overline{\sigma} + \alpha_1 \wedge \rho + \overline{\alpha}_1 \wedge \rho + \\
& + V_1\, \sigma \wedge \overline{\sigma} + V_2\, \sigma \wedge \rho + V_3\, \sigma \wedge \zeta + V_4\, \sigma \wedge \overline{\zeta} + \\
& + \overline{V_2}\, \overline{\sigma} \wedge \rho + \overline{V_4}\, \overline{\sigma} \wedge \zeta + \overline{V_3}\, \overline{\sigma} \wedge \overline{\zeta} + \\
& + V_8\, \rho \wedge \zeta + \overline{V_8}\, \rho \wedge \overline{\zeta} + \\
& + i\, \zeta \wedge \overline{\zeta},
\end{aligned}
$$

$$
\begin{aligned}
d\zeta = {} & \alpha_3 \wedge \sigma + \alpha_4 \wedge \overline{\sigma} + \alpha_5 \wedge \rho + \alpha_1 \wedge \zeta + \\
& + W_1\, \sigma \wedge \overline{\sigma} + W_2\, \sigma \wedge \rho + W_3\, \sigma \wedge \zeta + W_4\, \sigma \wedge \overline{\zeta} + \\
& + W_5\, \overline{\sigma} \wedge \rho + W_6\, \overline{\sigma} \wedge \zeta + W_7\, \overline{\sigma} \wedge \overline{\zeta} + \\
& + W_8\, \rho \wedge \zeta + W_9\, \rho \wedge \overline{\zeta} + \\
& + W_{10}\, \zeta \wedge \overline{\zeta}.
\end{aligned}
$$

- **Initial torsion coefficients:**

□ For $d\sigma$:

$$
\begin{aligned}
U_1 = {} & -\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\, K - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\, F + \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\, Q - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\, Q + \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\, B - \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^3}\, B + \frac{\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\, \overline{G} - \\
& - \frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\, B + \frac{\mathsf{e}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\, B - \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\, \overline{R} + \frac{\mathsf{d}}{\mathsf{a}\overline{\mathsf{a}}^3}\, \overline{R} + \frac{\mathsf{c}\overline{\mathsf{d}}}{\mathsf{a}^3\overline{\mathsf{a}}^3} - \frac{\overline{\mathsf{c}}\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}^3},
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= \frac{1}{a\bar{a}}\,F - \frac{b}{a^2\bar{a}}\,Q - \frac{\bar{b}}{a\bar{a}^2}\,B + \frac{e}{a^2\bar{a}},\\
U_3 &= \frac{1}{a}\,Q - \frac{c}{a^2\bar{a}},\\
U_4 &= \frac{1}{\bar{a}}\,B,
\end{aligned}
$$

$$
\begin{aligned}
U_5 &= \frac{1}{\bar{a}^2}\,\overline{G} - \frac{b}{a\bar{a}^2}\,B - \frac{\bar{b}}{\bar{a}^3}\,\overline{R} + \frac{\bar{d}}{a\bar{a}^2},\\
U_6 &= \frac{1}{\bar{a}}\,B - \frac{\bar{c}}{a\bar{a}^2},\\
U_7 &= \frac{a}{\bar{a}^2}\,\overline{R}.
\end{aligned}
$$

- For $d\rho$:

$$
\begin{aligned}
V_1 = &- \frac{c}{a^3\bar{a}^3}\,K - \frac{c\bar{c}}{a^4\bar{a}^4}\,F + \frac{bc\bar{c}}{a^5\bar{a}^4}\,Q - \frac{c\bar{d}}{a^4\bar{a}^3}\,Q + \frac{\bar{b}c\bar{c}}{a^4\bar{a}^5}\,B - \frac{c\bar{e}}{a^3\bar{a}^4}\,B + \frac{cc}{a^4\bar{a}^4}\,\overline{G} -\\
&- \frac{bcc}{a^5\bar{a}^4}\,B + \frac{ce}{a^4\bar{a}^3}\,B - \frac{\bar{b}cc}{a^4\bar{a}^5}\,\overline{R} + \frac{cd}{a^3\bar{a}^4}\,\overline{R} + \frac{cc\bar{d}}{a^5\bar{a}^4} - \frac{ec\bar{c}}{a^5\bar{a}^4} +\\
&+ \frac{\bar{c}}{a^3\bar{a}^3}\,\overline{K} + \frac{c\bar{c}}{a^4\bar{a}^4}\,\overline{F} - \frac{\bar{b}c\bar{c}}{a^4\bar{a}^5}\,\overline{Q} + \frac{\bar{c}d}{a^3\bar{a}^4}\,\overline{Q} - \frac{bc\bar{c}}{a^5\bar{a}^4}\,\overline{B} + \frac{\bar{c}e}{a^4\bar{a}^3}\,\overline{B} - \frac{\overline{cc}}{a^4\bar{a}^4}\,G +\\
&+ \frac{\bar{b}\overline{cc}}{a^4\bar{a}^5}\,\overline{B} - \frac{\overline{ce}}{a^3\bar{a}^4}\,\overline{B} + \frac{b\overline{cc}}{a^5\bar{a}^4}\,R - \frac{\bar{c}\bar{d}}{a^4\bar{a}^3}\,R - \frac{\overline{cc}d}{a^4\bar{a}^5} + \frac{c\overline{ce}}{a^4\bar{a}^5} -\\
&- i\,\frac{1}{a^2\bar{a}^2}\,J - \frac{\bar{c}}{a^3\bar{a}^3}\,E + \frac{b\bar{c}}{a^4\bar{a}^3}\,P - \frac{\bar{d}}{a^3\bar{a}^2}\,P + \frac{\bar{b}\bar{c}}{a^3\bar{a}^4}\,A - \frac{\bar{e}}{a^2\bar{a}^3}\,A + \frac{c}{a^3\bar{a}^3}\,\overline{E} - \frac{\bar{b}c}{a^3\bar{a}^4}\,\overline{P} + \frac{d}{a^2\bar{a}^3}\,\overline{P} -\\
&- \frac{bc}{a^4\bar{a}^3}\,A + \frac{e}{a^3\bar{a}^2}\,A - i\,\frac{bc\bar{e}}{a^4\bar{a}^4} - i\,\frac{\bar{b}\bar{c}e}{a^4\bar{a}^4} + i\,\frac{e\bar{e}}{a^3\bar{a}^3} + i\,\frac{b\bar{c}d}{a^4\bar{a}^4} + i\,\frac{\bar{b}c\bar{d}}{a^4\bar{a}^4} - i\,\frac{d\bar{d}}{a^3\bar{a}^3},
\end{aligned}
$$

$$V_2 = \frac{c}{a^3\bar a^2}\,F - \frac{bc}{a^4\bar a^2}\,Q - \frac{\bar b c}{a^3\bar a^3}\,B + \frac{ce}{a^4\bar a^2} + \frac{\bar c}{a^3\bar a^2}\,G - \frac{\bar b\bar c}{a^3\bar a^3}\,\overline{B} - \frac{b\bar c}{a^4\bar a^2}\,R + \frac{\bar c d}{a^3\bar a^3} + $$
$$+ \frac{1}{a^2\bar a}\,E - \frac{b}{a^3\bar a}\,P - \frac{\bar b}{a^2\bar a^2}\,A + i\,\frac{\bar b e}{a^3\bar a^2} - i\,\frac{bd}{a^3\bar a^2},$$
$$V_3 = \frac{c}{a^3\bar a}\,Q - \frac{cc}{a^4\bar a^2} + \frac{\bar c}{a^3\bar a}\,R + \frac{1}{a^2}\,P - i\,\frac{\bar b c}{a^3\bar a^2} + i\,\frac{d}{a^2\bar a},$$
$$V_4 = \frac{c}{a^2\bar a^2}\,B + \frac{\bar c}{a^2\bar a^2}\,\overline{B} - \frac{c\bar c}{a^3\bar a^3} + \frac{1}{a\bar a}\,A + i\,\frac{bc}{a^3\bar a^2} - i\,\frac{e}{a^2\bar a},$$
$$V_8 = \frac{c}{a^2\bar a} + i\,\frac{\bar b}{a\bar a}.$$

□ For $d\zeta$:

$$W_1 = -\frac{e}{a^3\bar a^3}\,K - \frac{\bar c e}{a^4\bar a^4}\,F + \frac{b\bar c e}{a^5\bar a^4}\,Q - \frac{\bar d e}{a^4\bar a^3}\,Q + \frac{\bar b\bar c e}{a^4\bar a^5}\,B - \frac{e\bar e}{a^3\bar a^4}\,B + \frac{ce}{a^4\bar a^4}\,\overline{G} -$$
$$- \frac{bce}{a^5\bar a^4}\,B + \frac{ee}{a^4\bar a^3}\,B - \frac{\bar b ce}{a^4\bar a^5}\,\overline{R} + \frac{de}{a^3\bar a^4}\,\overline{R} + \frac{c\bar d e}{a^5\bar a^4} - \frac{\bar c ee}{a^5\bar a^4} +$$
$$+ \frac{\bar d}{a^3\bar a^3}\,\overline{K} + \frac{c\bar d}{a^4\bar a^4}\,\overline{F} - \frac{\bar b c\bar d}{a^4\bar a^5}\,\overline{Q} + \frac{d\bar d}{a^3\bar a^4}\,\overline{Q} - \frac{bc\bar d}{a^5\bar a^4}\,\overline{B} + \frac{\bar d e}{a^4\bar a^3}\,\overline{B} - \frac{\bar c\bar d}{a^4\bar a^4}\,G +$$
$$+ \frac{\bar b\bar c\bar d}{a^4\bar a^5}\,\overline{B} - \frac{\bar d\bar e}{a^3\bar a^4}\,\overline{B} + \frac{b\bar c\bar d}{a^5\bar a^4}\,R - \frac{\bar d\bar d}{a^4\bar a^3}\,R - \frac{\bar c d\bar d}{a^4\bar a^5} + \frac{c\bar d\bar e}{a^4\bar a^5} -$$
$$- i\,\frac{b}{a^3\bar a^3}\,J - \frac{b\bar c}{a^4\bar a^4}\,E + \frac{bb\bar c}{a^5\bar a^4}\,P - \frac{b\bar d}{a^4\bar a^3}\,P + \frac{b\bar b\bar c}{a^4\bar a^5}\,A - \frac{b\bar e}{a^3\bar a^4}\,A + \frac{bc}{a^4\bar a^4}\,\overline{E} - \frac{b\bar b c}{a^4\bar a^5}\,\overline{P} + \frac{bd}{a^3\bar a^4}\,\overline{P} -$$
$$- \frac{bbc}{a^5\bar a^4}\,A + \frac{be}{a^4\bar a^3}\,A - i\,\frac{bbc\bar e}{a^5\bar a^5} - i\,\frac{b\bar b\bar c e}{a^5\bar a^5} + i\,\frac{be\bar e}{a^4\bar a^4} + i\,\frac{bb\bar c d}{a^5\bar a^5} + i\,\frac{b\bar b c\bar d}{a^5\bar a^5} - i\,\frac{bd\bar d}{a^4\bar a^4},$$

$$W_2 = \frac{e}{a^3\bar{a}^2}\,F - \frac{be}{a^4\bar{a}^2}\,Q - \frac{\bar{b}e}{a^3\bar{a}^3}\,B + \frac{ee}{a^4\bar{a}^2} + \frac{\bar{d}}{a^3\bar{a}^2}\,G - \frac{\overline{bd}}{a^3\bar{a}^3}\,\overline{B} - \frac{b\bar{d}}{a^4\bar{a}^2}\,R + \frac{d\bar{d}}{a^3\bar{a}^3} +$$

$$+ \frac{b}{a^3\bar{a}^2}\,E - \frac{bb}{a^4\bar{a}^2}\,P - \frac{b\bar{b}}{a^3\bar{a}^3}\,A + i\,\frac{b\bar{b}e}{a^4\bar{a}^3} - i\,\frac{bbd}{a^4\bar{a}^3},$$

$$W_3 = \frac{e}{a^3\bar{a}}\,Q - \frac{ce}{a^4\bar{a}^2} + \frac{\bar{d}}{a^3\bar{a}}\,R + \frac{b}{a^3\bar{a}}\,P - i\,\frac{b\bar{b}c}{a^4\bar{a}^3} + i\,\frac{bd}{a^3\bar{a}^2},$$

$$W_4 = \frac{e}{a^2\bar{a}^2}\,B + \frac{\bar{d}}{a^2\bar{a}^2}\,\overline{B} - \frac{c\bar{d}}{a^3\bar{a}^3} + \frac{b}{a^2\bar{a}^2}\,A + i\,\frac{bbc}{a^4\bar{a}^3} - i\,\frac{be}{a^3\bar{a}^2},$$

$$W_5 = \frac{e}{a^2\bar{a}^3}\,\overline{G} - \frac{be}{a^3\bar{a}^3}\,B - \frac{\bar{b}e}{a^2\bar{a}^4}\,\overline{R} + \frac{\bar{d}e}{a^3\bar{a}^3} + \frac{\bar{d}}{a^2\bar{a}^3}\,\overline{F} - \frac{\overline{bd}}{a^2\bar{a}^4}\,\overline{\mathsf{Q}} - \frac{b\bar{d}}{a^3\bar{a}^3}\,\overline{B} + \frac{\bar{e}\bar{d}}{a^2\bar{a}^4} +$$

$$+ \frac{b}{a^2\bar{a}^3}\,\overline{E} - \frac{b\bar{b}}{a^2\bar{a}^4}\,\overline{P} - \frac{bb}{a^3\bar{a}^3}\,\overline{A} - i\,\frac{bb\bar{e}}{a^3\bar{a}^4} + i\,\frac{b\overline{bd}}{a^3\bar{a}^4},$$

$$W_6 = \frac{e}{a^2\bar{a}^2}\,B - \frac{\bar{c}e}{a^3\bar{a}^3} + \frac{\bar{d}}{a^2\bar{a}^2}\,\overline{B} + \frac{b}{a^2\bar{a}^2}\,A - i\,\frac{b\bar{b}\bar{c}}{a^3\bar{a}^4} + i\,\frac{b\bar{e}}{a^2\bar{a}^3},$$

$$W_7 = \frac{e}{a\bar{a}^3}\,\overline{R} + \frac{\bar{d}}{a\bar{a}^3}\,\overline{Q} - \frac{\bar{c}\bar{d}}{a^2\bar{a}^4} + \frac{b}{a\bar{a}^3}\,\overline{P} + i\,\frac{bb\bar{c}}{a^3\bar{a}^4} - i\,\frac{b\bar{d}}{a^2\bar{a}^3},$$

$$W_8 = \frac{e}{a^2\bar{a}} + i\,\frac{b\bar{b}}{a^2\bar{a}^2},$$

$$W_9 = \frac{\bar{d}}{a\bar{a}^2} - i\,\frac{bb}{a^2\bar{a}^2},$$

$$W_{10} = i\,\frac{b}{a\bar{a}}.$$

• **Obtain 5 essential torsions potentially leading to normalizations of group parameters:**

$$
\boxed{\begin{aligned}
U_5 &= \frac{1}{\bar{\mathsf{a}}^2}\,\overline{G} - \frac{\mathsf{b}}{\mathsf{a}\bar{\mathsf{a}}^2}\,B - \frac{\bar{\mathsf{b}}}{\bar{\mathsf{a}}^3}\,\overline{R} + \frac{\bar{\mathsf{d}}}{\mathsf{a}\bar{\mathsf{a}}^2},\\
U_6 &= \frac{1}{\bar{\mathsf{a}}}\,B - \frac{\bar{\mathsf{c}}}{\mathsf{a}\bar{\mathsf{a}}^2},\\
U_7 &= \frac{\mathsf{a}}{\bar{\mathsf{a}}^2}\,\overline{R},\\
U_3 + \overline{U}_4 - 3\,V_8 &= \frac{1}{\mathsf{a}}\,Q - 4\,\frac{\mathsf{c}}{\mathsf{a}^2\bar{\mathsf{a}}} + \frac{1}{\mathsf{a}}\,\overline{B} - 3i\,\frac{\bar{\mathsf{b}}}{\mathsf{a}\bar{\mathsf{a}}},\\
\overline{U}_4 - V_8 - \overline{W}_{10} &= \frac{1}{\mathsf{a}}\,\overline{B} - \frac{\mathsf{c}}{\mathsf{a}\bar{\mathsf{a}}}.
\end{aligned}}
$$

• **Exceptional torsion:**

$$
\frac{\mathsf{a}}{\bar{\mathsf{a}}^2}\,\overline{R}.
$$

Proposition *When $R \not\equiv 0$, the coframe reduces to an $\{e\}$-structure on the base M — end of explorations.*

• **Remaining branch:**

$$
\boxed{\text{Suppose } R \equiv 0}\,.
$$

- **Three normalizations:**

$$\boxed{\mathsf{c} := \mathsf{a}\overline{\mathsf{a}}\,\overline{B}}\,.$$

$$\boxed{\mathsf{b} := \mathsf{a}\left(-\,i\,B + \tfrac{i}{3}\,\overline{Q}\right).}$$

$$\boxed{\mathsf{d} = \overline{\mathsf{a}}\left(-\;i\,\mathscr{L}\big(\overline{B}\big) + i\,P + \tfrac{2i}{3}\,\overline{B}\,Q\right).}$$

Lemma. *The exterior differential:*

$$\begin{aligned} d\mathbf{G}_0 = \mathscr{S}(\mathbf{G}_0)\cdot\sigma_0 + \overline{\mathscr{S}}(\mathbf{G}_0)\cdot\overline{\sigma}_0 + \mathscr{T}(\mathbf{G}_0)\cdot\rho_0 + \\ + \mathscr{L}(\mathbf{G}_0)\cdot\zeta_0 + \overline{\mathscr{L}}(\mathbf{G}_0)\cdot\overline{\zeta}_0 \end{aligned}$$

of an analytic function G *on* $M^5 \subset \mathbb{C}^4$ *expresses, in the lifted coframe, as:*

$$
\begin{aligned}
d\mathbf{G}_0 = \sigma \cdot \bigg(& \frac{1}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{S}(\mathbf{G}_0) - \frac{\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^2}\,\mathscr{T}(\mathbf{G}_0) + \frac{\mathsf{b}\mathsf{c}}{\mathsf{a}^4\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{G}_0) - \\
& - \frac{\mathsf{e}}{\mathsf{a}^3\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{G}_0) + \frac{\overline{\mathsf{b}}\mathsf{c}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{G}_0) - \frac{\mathsf{d}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
& + \overline{\sigma} \cdot \bigg(\frac{1}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{S}}(\mathbf{G}_0) - \frac{\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^3}\,\mathscr{T}(\mathbf{G}_0) + \frac{\mathsf{b}\overline{\mathsf{c}}}{\mathsf{a}^3\overline{\mathsf{a}}^3}\,\mathscr{L}(\mathbf{G}_0) - \\
& - \frac{\overline{\mathsf{d}}}{\mathsf{a}^2\overline{\mathsf{a}}^2}\,\mathscr{L}(\mathbf{G}_0) + \frac{\overline{\mathsf{b}}\overline{\mathsf{c}}}{\mathsf{a}^2\overline{\mathsf{a}}^4}\,\overline{\mathscr{L}}(\mathbf{G}_0) - \frac{\overline{\mathsf{e}}}{\mathsf{a}\overline{\mathsf{a}}^3}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
& + \rho \cdot \bigg(\frac{1}{\mathsf{a}\overline{\mathsf{a}}}\,\mathscr{T}(\mathbf{G}_0) - \frac{\mathsf{b}}{\mathsf{a}^2\overline{\mathsf{a}}}\,\mathscr{L}(\mathbf{G}_0) - \frac{\overline{\mathsf{b}}}{\mathsf{a}\overline{\mathsf{a}}^2}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg) + \\
& + \zeta \cdot \bigg(\frac{1}{\mathsf{a}}\,\mathscr{L}(\mathbf{G}_0)\bigg) + \\
& + \overline{\zeta} \cdot \bigg(\frac{1}{\overline{\mathsf{a}}}\,\overline{\mathscr{L}}(\mathbf{G}_0)\bigg).
\end{aligned}
$$

- **Final normalization:**

$$
\boxed{\mathsf{e} := \mathsf{a} \cdot \big(i\,\overline{\mathscr{L}}(\overline{B}) - i\,A - 2i\,B\overline{B} + \tfrac{i}{3}\,\overline{B}\,\overline{Q}\big).}
$$

- **Examine other subranches which potentially normalize** a**.**

Cartan Connections in CR Geometry

• **Homogeneous space:** Beloshapka's cubic model $M_{\mathsf{c}}^5 \subset \mathbb{C}^4$ happens to be a *homogeneous space*:

$$M_{\mathsf{c}}^5 \cong G^7 / H^2 \cong N_4^5,$$

namely a quotient of a 7-dimensional real Lie group G^7 by a 2-dimensional closed commutative Lie subgroup $N^2 \cong (\mathbb{C}^*, \times)$, the resulting quotient being the unique connected and simply connected *nilpotent* Lie group corresponding to the real *nilpotent* Lie algebra with generators x_1, x_2, x_3, x_4, x_5 named $\mathfrak{n}_5^4$ in the Goze-Remm classification:

$$\mathfrak{n}_5^4: \qquad \begin{cases} [\mathsf{x}_1, \mathsf{x}_2] = \mathsf{x}_3, \\ [\mathsf{x}_1, \mathsf{x}_3] = \mathsf{x}_4, \\ [\mathsf{x}_2, \mathsf{x}_3] = \mathsf{x}_5, \end{cases}$$

unwritten brackets being zero.

• **Infinitesimal CR automorphisms** The Lie algebra:

$$\mathfrak{aut}_{\mathrm{CR}}(M_{\mathsf{c}}^5)$$

of infinitesimal CR automorphisms of Beloshapka's cubic model $M^5_{\mathsf{c}} \subset \mathbb{C}^4$ is 7-dimensional, generated by the real parts of the following 7 vector fields of type $(1,0)$ with holomorphic coefficients:

$$
\begin{aligned}
S_2 &:= \frac{\partial}{\partial w_3}, \\
S_1 &:= \frac{\partial}{\partial w_2}, \\
T &:= \frac{\partial}{\partial w_1}, \\
L_1 &:= \frac{\partial}{\partial z} + 2\sqrt{-1}\, z\, \frac{\partial}{\partial w_1} + \big(2\sqrt{-1}\, z^2 + 4\, w_1\big)\, \frac{\partial}{\partial w_2} + 2\, z^2\, \frac{\partial}{\partial w_3}, \\
L_2 &:= \sqrt{-1}\, \frac{\partial}{\partial z} + 2\, z\, \frac{\partial}{\partial w_1} + 2\, z^2\, \frac{\partial}{\partial w_2} - \big(2\sqrt{-1}\, z^2 - 4\, w_1\big)\, \frac{\partial}{\partial w_3}, \\
D &:= z\, \frac{\partial}{\partial z} + 2\, w_1\, \frac{\partial}{\partial w_1} + 3\, w_2\, \frac{\partial}{\partial w_2} + 3\, w_3\, \frac{\partial}{\partial w_3}, \\
R &:= \sqrt{-1}\, z\, \frac{\partial}{\partial z} - w_3\, \frac{\partial}{\partial w_2} + w_2\, \frac{\partial}{\partial w_3}.
\end{aligned}
$$

- **Isotropy subalgebra of the origin:**

$$
\mathbb{R}\big(D + \overline{D}\big) \oplus \mathbb{R}\big(R + \overline{R}\big).
$$

- **Commutator table:**

	S_2	S_1	T	L_2	L_1	D	R
S_2	0	0	0	0	0	$3S_2$	$-S_1$
S_1	$*$	0	0	0	0	$3S_1$	S_2
T	$*$	$*$	0	$4S_2$	$4S_1$	$2T$	0
L_2	$*$	$*$	$*$	0	$-4T$	L_2	$-L_1$
L_1	$*$	$*$	$*$	$*$	0	L_1	L_2
D	$*$	$*$	$*$	$*$	$*$	0	0
R	$*$	$*$	$*$	$*$	$*$	$*$	0

• **Nilpotent subalgebra:** The five fields S_2, S_1, T, L_2, L_1 generate a nilpotent Lie subalgebra of $\mathfrak{aut}_{\rm CR}(M^5_{\sf c})$ visibly isomorphic to $\mathfrak{n}^4_5$.

Theorem. [Merker-Pocchiola-Sabzevari] *Associated to every Class* III_1 *local CR-generic* $\mathscr{C}^\omega$ *submanifold* $M^5 \subset \mathbb{C}^4$*, there is a canonical Cartan connection* (P^7, ϖ) *modelled on the nilpotent homogeneous space* $G^7/H^2 \cong N^5_4 \cong M^5_{\sf c}$ *whose natural orbit space is Beloshapka's cubic model* $M^5_{\sf c} \subset \mathbb{C}^4$*.*

CR-umbilics of Real Ellipsoids

Despite their importance, until now, the invariants of pseudoconvex domains have been fully completed, to our knowledge, only in the case of the unit ball $\mathbb{B}^{n+1} \subset \mathbb{C}^{n+1}$, where they all vanish! Webster 2000

- **Coordinates on $\mathbb{C}^{\mathrm{N}\geqslant 2} \cong \mathbb{R}^{2\mathrm{N}\geqslant 4}$:**

$$\boldsymbol{z}_i = \boldsymbol{x}_i + \sqrt{-1}\,\boldsymbol{y}_i \qquad (1 \leqslant i \leqslant \mathrm{N}).$$

- **Ellipsoid:**

$$(\mathsf{E}_{\alpha,\beta}) \qquad \sum_{1\leqslant i\leqslant \mathrm{N}} \big(a_i\,\boldsymbol{x}_i^2 + b_i\,\boldsymbol{y}_i^2\big) = 1,$$

with real constants $a_i \geqslant b_i > 0$.

- **Alternative view:** [Webster 2000]

$$(\mathsf{E}_{A_1,\dots,A_\mathrm{N}}) \qquad \sum_{1\leqslant i\leqslant \mathrm{N}} \big(\boldsymbol{z}_i\overline{\boldsymbol{z}}_i + A_i\,(\boldsymbol{z}_i^2 + \overline{\boldsymbol{z}}_i^2)\big) = 1.$$

- **CR-Umbilical locus:**

$$\mathsf{Umb}_{\mathsf{CR}}\big(E_{A_1,\dots,A_{\rm N}}\big) := \Big\{p \in E_{A_1,\dots,A_{\rm N}} \colon\ S_{\alpha\rho}^{\beta\sigma}(p) = 0,\ \forall\,\alpha,\rho,\beta,\sigma\Big\}.$$

Theorem. [Webster 2000] *In* $\mathbb{C}^{\rm N \geqslant 3}$*, if* $0 < A_1 < \cdots < A_{\rm N} < \frac{1}{2}$*, then*

$$\emptyset = \mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{A_1,\dots,A_{\rm N}}\big). \qquad \square$$

- **Observation:** In $\mathbb{C}^2$:

$$S_{\alpha\rho}^{\beta\sigma} = S_{1,1}^{1,1}.$$

Theorem. [Huang-Ji 2007] *Every ellipsoid* $E_{a_1,a_2} \subset \mathbb{C}^2$ *has at least* **4** *CR-umbilical points.*

- **Question:** *Do compact connected* $\mathscr{C}^\omega$ *Levi nondegenerate hypersurfaces have all CR-umbilical points?*

- **Prior question:** *In* $\mathbb{C}^2$*, how the Cartan CR invariant:*

$$S_{11}^{11} \doteqdot \mathfrak{J}_{\mathsf{Cartan}}$$

looks like?

• **Hypersurface:**

$$M^3 \subset \mathbb{C}^2,$$

of class $\mathscr{C}^\omega$, $\mathscr{C}^\infty$, $\mathscr{C}^6$, Levi nondegenerate.

• **Implicit:**

$$M = \Big\{(z,w) \in \mathbb{C}^2 \colon \ \rho\big(z,w,\overline{z},\overline{w}\big) = 0\Big\}.$$

• **Reality:**

$$\overline{\rho} = \rho.$$

• **Smoothness:** At each $p \in M$:

$$\rho_z(p) \neq 0 \qquad\qquad \text{or} \qquad\qquad \rho_w(p) \neq 0.$$

• **CR vector fields of type $(1,0)$ and $(0,1)$:**

$$\mathscr{L} := -\rho_w \frac{\partial}{\partial z} + \rho_z \frac{\partial}{\partial w} \qquad\qquad \text{and} \qquad\qquad \overline{\mathscr{L}} := -\rho_{\overline{w}} \frac{\partial}{\partial \overline{z}} + \rho_{\overline{z}} \frac{\partial}{\partial \overline{w}}.$$

• **Levi nondegeneracy:**

$$\big[\mathscr{L}, \overline{\mathscr{L}}\big] \not\equiv 0 \ \ \mathsf{mod} \big(\mathscr{L} \oplus \overline{\mathscr{L}}\big).$$

• **Explicitly:**

$$0 \neq - \begin{vmatrix} 0 & \rho_z & \rho_w \\ \rho_{\overline{z}} & \rho_{z\overline{z}} & \rho_{w\overline{z}} \\ \rho_{\overline{w}} & \rho_{z\overline{w}} & \rho_{w\overline{w}} \end{vmatrix}$$

$$= \rho_{\overline{z}}\rho_z\rho_{w\overline{w}} - \rho_{\overline{z}}\rho_w\rho_{z\overline{w}} - \rho_{\overline{w}}\rho_z\rho_{\overline{z}w} + \rho_{\overline{w}}\rho_w\rho_{z\overline{z}}.$$

- **Complex Hessian:**

$$\mathsf{H}(\rho) := \begin{pmatrix} -\rho_w & \rho_z \end{pmatrix} \begin{pmatrix} \rho_{zz} & \rho_{wz} \\ \rho_{zw} & \rho_{ww} \end{pmatrix} \begin{pmatrix} -\rho_w \\ \rho_z \end{pmatrix}$$

$$= -\rho_w\rho_w\rho_{zz} + 2\rho_z\rho_w\rho_{zw} - \rho_z\rho_z\rho_{ww}.$$

Lemma. [Son et al. 2016] *The Cartan CR invariant is a nonzero multiple of the determinant:*

$$\mathfrak{J}_{\mathsf{Cartan}} = \mathsf{nonzero} \cdot \begin{vmatrix} \rho_w^3 & \overline{\mathscr{L}}(\rho_w^3) & \overline{\mathscr{L}}^2(\rho_w^3) & \overline{\mathscr{L}}^3(\rho_w^3) & \overline{\mathscr{L}}^4(\rho_w^3) \\ \rho_z\rho_w^2 & \overline{\mathscr{L}}(\rho_z\rho_w^2) & \overline{\mathscr{L}}^2(\rho_z\rho_w^2) & \overline{\mathscr{L}}^3(\rho_z\rho_w^2) & \overline{\mathscr{L}}^4(\rho_z\rho_w^2) \\ \rho_z^2\rho_w & \overline{\mathscr{L}}(\rho_z^2\rho_w) & \overline{\mathscr{L}}^2(\rho_z^2\rho_w) & \overline{\mathscr{L}}^3(\rho_z^2\rho_w) & \overline{\mathscr{L}}^4(\rho_z^2\rho_w) \\ \rho_z^3 & \overline{\mathscr{L}}(\rho_z^3) & \overline{\mathscr{L}}^2(\rho_z^3) & \overline{\mathscr{L}}^3(\rho_z^3) & \overline{\mathscr{L}}^4(\rho_z^3) \\ \mathsf{H}(\rho) & \overline{\mathscr{L}}(\mathsf{H}(\rho)) & \overline{\mathscr{L}}^2(\mathsf{H}(\rho)) & \overline{\mathscr{L}}^3(\mathsf{H}(\rho)) & \overline{\mathscr{L}}^4(\mathsf{H}(\rho)) \end{vmatrix}.$$

Corollary. [Merker 2017] *In real representation, the number of differential monomials in the pure* 6*-jet:*

$$J^6_{x,y,u,v}\rho(x,y,u,v) \;=\; \mathbf{419}\,\text{variables}$$

of the respective $\mathbf{5}\times\mathbf{5}=\mathbf{25}$ *entries is:*

$$\begin{vmatrix} \mathbf{4} & \mathbf{28} & \mathbf{232} & \mathbf{1\,760} & \mathbf{11\,772} \\ \mathbf{6} & \mathbf{40} & \mathbf{346} & \mathbf{2\,396} & \mathbf{15\,298} \\ \mathbf{6} & \mathbf{40} & \mathbf{346} & \mathbf{2\,396} & \mathbf{15\,298} \\ \mathbf{4} & \mathbf{28} & \mathbf{232} & \mathbf{1\,760} & \mathbf{11\,772} \\ \mathbf{34} & \mathbf{184} & \mathbf{1\,389} & \mathbf{9\,392} & \mathbf{56\,745} \end{vmatrix}.$$

• **Computational obstacle:** This determinant is not expandable, since for instance:

$$\mathbf{34}\cdot\mathbf{28}\cdot\mathbf{346}\cdot\mathbf{2\,396}\cdot\mathbf{11\,772} \;=\; \mathbf{9\,290\,735\,887\,104}.$$

• **Compare with:** [M.-Sabzevari 2012]

$$\sim\;4\text{ hours of computations}\;\longleftrightarrow\;\sim\;\mathbf{1\,500\,000}\,\text{terms}.$$

• **Universal Phenomenon in True Algebra:** *Initial expressions share entangled relations and hide serendipitous compactifications.*

• **Question:** *For an ellipsoid E in $\mathbb{C}^2$, can $\mathsf{Umb}_{\mathsf{CR}}(E)$ be described explicity?*

Theorem. [Foo-Merker-Ta 2017] *For every real numbers $a \geqslant 1$, $b \geqslant 1$ with $(a,b) \neq (1,1)$, the curve parametrized by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\gamma\colon \quad \theta \longmapsto \big(x(\theta) + \sqrt{-1}\,y(\theta),\ u(\theta) + \sqrt{-1}\,v(\theta)\big)$$

with components:

$$x(\theta) := \sqrt{\frac{a-1}{a\,(ab-1)}}\,\cos\theta, \qquad y(\theta) := \sqrt{\frac{b\,(a-1)}{ab-1}}\,\sin\theta,$$

$$u(\theta) := \sqrt{\frac{b-1}{b\,(ab-1)}}\,\sin\theta, \qquad v(\theta) := -\sqrt{\frac{a\,(b-1)}{ab-1}}\,\cos\theta,$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{a,b}\big) \subset \mathsf{E}_{a,b}$$

of the ellipsoid $\mathsf{E}_{a,b} \subset \mathbb{C}^2$ of equation $a\,x^2 + y^2 + b\,u^2 + y^2 = 1$.

• **Difficulty:** Deal with the single Cartan CR invariant of $M = \{\rho = 0\}$:

$$\mathfrak{I}_{\mathsf{Cartan}} = \mathfrak{I}_{\mathsf{Cartan}}(\rho),$$

which contains:

$$\textbf{38 310} \text{ differential monomials in } J^{\mathbf{6}}_{x,y,u,v}\rho,$$

and show that:

$$0 \overset{?}{=} \gamma^*\big(\mathfrak{I}_{\mathsf{Cartan}}\big)(\theta) \qquad\qquad (\forall\,\theta\in\mathbb{R}).$$

- **Indispensable exigence:** Possess an appropriate formula for $\mathfrak{I}_{\mathsf{Cartan}}(\rho)$.

- **Serendipity:** Such a formula already exists [Merker 2010].

Theorem. [Foo-Merker-Ta 2017] *A point* $p \in M = \{\rho = 0\}$ *is CR-umbilical iff:*

$$0 = \boldsymbol{I}_{[w]}(p),$$

where:

$$\begin{aligned}
\boldsymbol{I}_{[w]} := \ & 12\,(\rho_w)^9 \Bigg\{ \bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^3 \overline{L}^4\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - 6\,\bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^3\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \\
& - 4\,\bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}^2\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^2\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \bigg[\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg]^2 \overline{L}^3\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) + \\
& + 15\,\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg[\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\bigg]^2 \overline{L}^2\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) + 10\,\frac{\mathsf{L}(\rho)}{\rho_w^2}\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}^2\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg) - \\
& - 15\,\bigg[\overline{L}\bigg(\frac{\mathsf{L}(\rho)}{\rho_w^2}\bigg)\bigg]^3 \overline{L}\bigg(\frac{\mathsf{H}(\rho)}{\rho_w^3}\bigg)\Bigg\}.
\end{aligned}$$

- **Exercise:** Show:

$$\boldsymbol{I}_{[w]} = \boldsymbol{I}_{[z]}.$$

- **Ellipsoids:**

$$0 = \rho = a\,x^2 + y^2 + b\,u^2 + v^2 - 1$$

- **Proof of the theorem:** Verify:

$$0 \overset{?}{=} \gamma^*\big(\boldsymbol{I}_{[w]}\big)(\theta) \qquad\qquad (\forall\,\theta \in \mathbb{R}).$$

Drop the factor $12\,(\rho_w)^9 \doteqdot 1$, and call $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, $\boldsymbol{T}_3$, $\boldsymbol{T}_4$, $\boldsymbol{T}_5$, $\boldsymbol{T}_6$, $\boldsymbol{T}_7$ the seven concerned terms, so that the goal becomes:

$$0 \overset{?}{=} \gamma^*(\boldsymbol{T}_1) + \gamma^*(\boldsymbol{T}_2) + \gamma^*(\boldsymbol{T}_3) + \gamma^*(\boldsymbol{T}_4) + \gamma^*(\boldsymbol{T}_5) + \gamma^*(\boldsymbol{T}_6) + \gamma^*(\boldsymbol{T}_7).$$

- **Hand-on-computer calculations:**

$$
\begin{aligned}
\boldsymbol{T}_1 &= \frac{1}{8}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_1}{\boldsymbol{D}}, \\
\boldsymbol{T}_2 &= \frac{3}{4}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_2}{\boldsymbol{D}}, \\
\boldsymbol{T}_3 &= \frac{1}{2}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_3}{\boldsymbol{D}}, \\
\boldsymbol{T}_4 &= \frac{1}{8}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_4}{\boldsymbol{D}}, \\
\boldsymbol{T}_5 &= \frac{15}{8}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_5}{\boldsymbol{D}}, \\
\boldsymbol{T}_6 &= \frac{5}{4}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_6}{\boldsymbol{D}}, \\
\boldsymbol{T}_7 &= \frac{15}{8}\sqrt{-1}\,(a-1)\,\frac{\boldsymbol{N}_7}{\boldsymbol{D}},
\end{aligned}
$$

with, in denominator place:

$$
\boldsymbol{D} := \left(\sqrt{a}\cos\theta - \sqrt{-1}\,\sqrt{b}\sin\theta\right)^{8}(a\,b-1)\left(\frac{b-1}{a\,b-1}\right)^{\frac{11}{2}},
$$

with numerator 1:

$$N_1 := \cos^7\theta\left[499\,a^{9/2}b^3 + 625\,a^{9/2}b^2 - 233\,a^{7/2}b^3 + 205\,a^{9/2}b - 631\,a^{7/2}b^2 + 15\,a^{9/2} - 415\,a^{7/2}b - 65\,a^{7/2}\right]$$
$$+\sqrt{-1}\cos^6\theta\sin\theta\left[2887\,a^4b^{7/2} + 4401\,a^4b^{5/2} - 1297\,a^3b^{7/2} + 1905\,a^4b^{3/2} - 4059\,a^3b^{5/2} + 215\,a^4b^{1/2} - 3327\,a^3b^{3/2} - 725\,a^3b^{1/2}\right]$$
$$+\cos^5\theta\sin^2\theta\left[-7023\,a^{7/2}b^4 - 13021\,a^{7/2}b^3 + 3013\,a^{5/2}b^4 - 7105\,a^{7/2}b^2 + 11011\,a^{5/2}b^3 - 1075\,a^{7/2}b + 11059\,a^{5/2}b^2 + 3141\,a^{5/2}b\right]$$
$$+\sqrt{-1}\cos^4\theta\sin^3\theta\left[-9267\,a^3b^{9/2} - 20989\,a^3b^{7/2} + 3757\,a^2b^{9/2} - 14101\,a^3\,b^{5/2} + 16279\,a^2b^{7/2} - 2683\,a^3b^{3/2} + 19891\,a^2b^{5/2} + 7113\,a^2b^{3/2}\right]$$
$$+\cos^3\theta\sin^4\theta\left[7113\,a^{5/2}b^5 + 19891\,a^{5/2}b^4 - 2683\,a^{3/2}b^5 + 16279\,a^{5/2}b^3 - 14101\,a^{3/2}b^4 + 3757\,a^{5/2}b^2 - 20989\,a^{3/2}b^3 - 9267\,a^{3/2}b^2\right]$$
$$+\sqrt{-1}\cos^2\theta\sin^5\theta\left[3141\,a^2b^{11/2} + 11059\,a^2b^{9/2} - 1075\,ab^{11/2} + 11011\,a^2b^{7/2} - 7105\,ab^{9/2} + 3013\,a^2b^{5/2} - 13021\,ab^{7/2} - 7023\,ab^{5/2}\right]$$
$$+\cos^1\theta\sin^6\theta\left[-725\,a^{3/2}b^6 - 3327\,a^{3/2}b^5 + 215\,a^{1/2}b^6 - 4059\,a^{3/2}b^4 + 1905\,a^{1/2}b^5 - 1297\,a^{3/2}b^3 + 4401\,a^{1/2}b^4 + 2287\,a^{1/2}b^3\right]$$
$$+\sqrt{-1}\sin^7\theta\left[-65\,ab^{13/2} - 415\,ab^{11/2} + 15\,b^{13/2} - 631\,ab^{9/2} + 205\,b^{11/2} - 233\,ab^{7/2} + 625\,b^{9/2} + 499\,b^{7/2}\right],$$

with numerator 2:

$$N_2 := \cos^7\theta\left[-165\,a^{9/2}b^3 - 193\,a^{9/2}\,b^2 + 93\,a^{7/2}b^3 - 67\,a^{9/2}b + 205\,a^{7/2}b^2 - 7\,a^{9/2} + 115\,a^{7/2}b + 19\,a^{7/2}\right]$$
$$+\sqrt{-1}\cos^6\theta\sin\theta\left[-925\,a^4b^{7/2} - 1389\,a^4b^{5/2} + 505\,a^3b^{7/2} - 627\,a^4b^{3/2} + 1341\,a^3b^{5/2} - 83\,a^4b^{1/2} + 975\,a^3b^{3/2} + 203\,a^3b^{1/2}\right]$$
$$+\cos^5\theta\sin^2\theta\left[2177\,a^{7/2}b^4 + 4141\,a^{7/2}b^3 - 1145\,a^{5/2}b^4 + 2359\,a^{7/2}b^2 - 3673\,a^{5/2}b^3 + 395\,a^{7/2}b - 3367\,a^{5/2}b^2 - 887\,a^{5/2}b\right]$$
$$+\sqrt{-1}\cos^4\theta\sin^3\theta\left[2777\,a^3b^{9/2} + 6649\,a^3b^{7/2} - 1397\,a^2b^{9/2} + 4711\,a^3b^{5/2} - 5449\,a^2b^{7/2} + 983\,a^3b^{3/2} - 6211\,a^2b^{5/2} - 2063\,a^2b^{3/2}\right]$$
$$+\cos^3\theta\sin^4\theta\left[-2063\,a^{5/2}b^5 - 6211\,a^{5/2}b^4 + 983\,a^{3/2}b^5 - 5449\,a^{5/2}b^3 + 4711\,a^{3/2}b^4 - 1397\,a^{5/2}b^2 + 6649\,a^{3/2}b^3 + 2777\,a^{3/2}b^2\right]$$
$$+\sqrt{-1}\cos^2\theta\sin^5\theta\left[-887\,a^2b^{11/2} - 3367\,a^2b^{9/2} + 395\,ab^{11/2} - 3673\,a^2b^{7/2} + 2359\,ab^{9/2} - 1145\,a^2b^{5/2} + 4141\,ab^{7/2} + 2177\,ab^{5/2}\right]$$
$$+\cos^1\theta\sin^6\theta\left[203\,a^{3/2}b^6 + 975\,a^{3/2}b^5 - 83\,a^{1/2}b^6 + 1341\,a^{3/2}b^4 - 627\,a^{1/2}b^5 + 505\,a^{3/2}b^3 - 1389\,a^{1/2}b^4 - 925\,a^{1/2}b^3\right]$$
$$+\sqrt{-1}\sin^7\theta\left[19\,ab^{13/2} + 115\,ab^{11/2} - 7\,b^{13/2} + 205\,ab^{9/2} - 67\,b^{11/2} + 93\,ab^{7/2} - 193\,b^{9/2} - 165\,b^{7/2}\right],$$

with numerator 3:

$$N_3 := \cos^7\theta\left[-91\,a^{9/2}b^3 - 109\,a^{9/2}b^2 + 65\,a^{7/2}b^3 - 37\,a^{9/2}b + 115\,a^{7/2}b^2 - 3\,a^{9/2} + 55\,a^{7/2}b + 5\,a^{7/2}\right]$$
$$+\sqrt{-1}\cos^6\theta\sin\theta\left[-499\,a^4b^{7/2} - 777\,a^4b^{5/2} + 349\,a^3b^{7/2} - 357\,a^4b^{3/2} + 771\,a^3b^{5/2} - 47\,a^4b^{1/2} + 483\,a^3b^{3/2} + 77\,a^3b^{1/2}\right]$$
$$+\cos^5\theta\sin^2\theta\left[1143\,a^{7/2}b^4 + 2281\,a^{7/2}b^3 - 781\,a^{5/2}b^4 + 1369\,a^{7/2}b^2 - 2143\,a^{5/2}b^3 + 247\,a^{7/2}b - 1723\,a^{5/2}b^2 - 393\,a^{5/2}b\right]$$
$$+\sqrt{-1}\cos^4\theta\sin^3\theta\left[1407\,a^3b^{9/2} + 3589\,a^3b^{7/2} - 937\,a^2b^{9/2} + 2761\,a^3b^{5/2} - 3199\,a^2b^{7/2} + 643\,a^3b^{3/2} - 3271\,a^2b^{5/2} - 993\,a^2b^{3/2}\right]$$
$$+\cos^3\theta\sin^4\theta\left[-993\,a^{5/2}b^5 - 3271\,a^{5/2}b^4 + 643\,a^{3/2}b^5 - 3199\,a^{5/2}b^3 + 2761\,a^{3/2}b^4 - 937\,a^{5/2}b^2 + 3589\,a^{3/2}b^3 + 1407\,a^{3/2}b^2\right]$$
$$+\sqrt{-1}\cos^2\theta\sin^5\theta\left[-393\,a^2b^{11/2} - 1723\,a^2b^{9/2} + 247\,ab^{11/2} - 2143\,a^2b^{7/2} + 1369\,ab^{9/2} - 781\,a^2b^{5/2} + 2281\,ab^{7/2} + 1143\,ab^{5/2}\right]$$
$$+\cos^1\theta\sin^6\theta\left[77\,a^{3/2}b^6 + 483\,a^{3/2}b^5 - 47\,a^{1/2}b^6 + 771\,a^{3/2}b^4 - 357\,a^{1/2}b^5 + 349\,a^{3/2}b^3 - 777\,a^{1/2}b^4 - 499\,a^{1/2}b^3\right]$$
$$+\sqrt{-1}\sin^7\theta\left[5\,ab^{13/2} + 55\,ab^{11/2} - 3\,b^{13/2} + 115\,ab^{9/2} - 37\,b^{11/2} + 65\,ab^{7/2} - 109\,b^{9/2} - 91\,b^{7/2}\right],$$

with numerator 4:

$$N_4 := \cos^7\theta \Big[-\, 75\, a^{9/2}b^3 - 91\, a^{9/2}b^2 + 75\, a^{7/2}b^3 - 25\, a^{9/2}b + 91\, a^{7/2}b^2 - a^{9/2} + 25\, a^{7/2}b + a^{7/2} \Big]$$
$$+ \sqrt{-1}\cos^6\theta \sin\theta \Big[-\, 391\, a^4b^{7/2} - 639\, a^4b^{5/2} + 391\, a^3b^{7/2} - 285\, a^4b^{3/2} + 639\, a^3b^{5/2} - 29\, a^4b^{1/2} + 285\, a^3b^{3/2} + 29\, a^3b^{1/2} \Big]$$
$$+ \cos^5\theta \sin^2\theta \Big[839\, a^{7/2}b^4 + 1831\, a^{7/2}b^3 - 839\, a^{5/2}b^4 + 1165\, a^{7/2}b^2 - 1831\, a^{5/2}b^3 + 197\, a^{7/2}b - 1165\, a^{5/2}b^2 - 197\, a^{5/2}b \Big]$$
$$+ \sqrt{-1}\cos^4\theta \sin^3\theta \Big[947\, a^3b^{9/2} + 2779\, a^3b^{7/2} - 947\, a^2b^{9/2} + 2401\, a^3b^{5/2} - 2779\, a^2b^{7/2} + 593\, a^3b^{3/2} - 2401\, a^2b^{5/2} - 593\, a^2b^{3/2} \Big]$$
$$+ \cos^3\theta \sin^4\theta \Big[-\, 593\, a^{5/2}b^5 - 2401\, a^{5/2}b^4 + 593\, a^{3/2}b^5 - 2779\, a^{5/2}b^3 + 2401\, a^{3/2}b^4 - 947\, a^{5/2}b^2 + 2779\, a^{3/2}b^3 + 947\, a^{3/2}b^2 \Big]$$
$$+ \sqrt{-1}\cos^2\theta \sin^5\theta \Big[-\, 197\, a^2b^{11/2} - 1165\, a^2b^{9/2} + 197\, ab^{11/2} - 1831\, a^2b^{7/2} + 1165\, ab^{9/2} - 839\, a^2b^{5/2} + 1831\, ab^{7/2} + 839\, ab^{5/2} \Big]$$
$$+ \cos^1\theta \sin^6\theta \Big[29\, a^{3/2}b^6 + 285\, a^{3/2}b^5 - 29\, a^{1/2}b^6 + 639\, a^{3/2}b^4 - 285\, a^{1/2}b^5 + 391\, a^{3/2}b^3 - 639\, a^{1/2}b^4 - 391\, a^{1/2}b^3 \Big]$$
$$+ \sqrt{-1}\sin^7\theta \Big[ab^{13/2} + 25\, ab^{11/2} - b^{13/2} + 91\, ab^{9/2} - 25\, b^{11/2} + 75\, ab^{7/2} - 91\, b^{9/2} - 75\, b^{7/2} \Big],$$

with numerator 5:

$$N_5 := \cos^7\theta \Big[63\, a^{9/2}b^3 + 69\, a^{9/2}b^2 - 45\, a^{7/2}b^3 + 25\, a^{9/2}b - 75\, a^{7/2}b^2 + 3\, a^{9/2} - 35\, a^{7/2}b - 5\, a^{7/2} \Big]$$
$$+ \sqrt{-1}\cos^6\theta \sin\theta \Big[339\, a^4b^{7/2} + 509\, a^4b^{5/2} - 237\, a^3b^{7/2} + 237\, a^4b^{3/2} - 511\, a^3b^{5/2} + 35\, a^4b^{1/2} - 315\, a^3b^{3/2} - 57\, a^3b^{1/2} \Big]$$
$$+ \cos^5\theta \sin^2\theta \Big[-\, 763\, a^{7/2}b^4 - 1521\, a^{7/2}b^3 + 521\, a^{5/2}b^4 - 909\, a^{7/2}b^2 + 1431\, a^{5/2}b^3 - 167\, a^{7/2}b + 1143\, a^{5/2}b^2 + 265\, a^{5/2}b \Big]$$
$$+ \sqrt{-1}\cos^4\theta \sin^3\theta \Big[-\, 927\, a^3b^{9/2} - 2409\, a^3b^{7/2} + 617\, a^2b^{9/2} - 1841\, a^3b^{5/2} + 2139\, a^2b^{7/2} - 423\, a^3b^{3/2} + 2191\, a^2b^{5/2} + 653\, a^2b^{3/2} \Big]$$
$$+ \cos^3\theta \sin^4\theta \Big[653\, a^{5/2}b^5 + 2191\, a^{5/2}b^4 - 423\, a^{3/2}b^5 + 2139\, a^{5/2}b^3 - 1841\, a^{3/2}b^4 + 617\, a^{5/2}b^2 - 2409\, a^{3/2}b^3 - 927\, a^{3/2}b^2 \Big]$$
$$+ \sqrt{-1}\cos^2\theta \sin^5\theta \Big[265\, a^2b^{11/2} + 1143\, a^2b^{9/2} - 167\, ab^{11/2} + 1431\, a^2b^{7/2} - 909\, ab^{9/2} + 521\, a^2b^{5/2} - 1521\, ab^{7/2} - 763\, ab^{5/2} \Big]$$
$$+ \cos^1\theta \sin^6\theta \Big[-\, 57\, a^{3/2}b^6 - 315\, a^{3/2}b^5 + 35\, a^{1/2}b^6 - 511\, a^{3/2}b^4 + 237\, a^{1/2}b^5 - 237\, a^{3/2}b^3 + 509\, a^{1/2}b^4 + 339\, a^{1/2}b^3 \Big]$$
$$+ \sqrt{-1}\sin^7\theta \Big[-\, 5\, ab^{13/2} - 35\, ab^{11/2} + 3\, b^{13/2} - 75\, ab^{9/2} + 25\, b^{11/2} - 45\, ab^{7/2} + 69\, b^{9/2} + 63\, b^{7/2} \Big],$$

with numerator 6:

$$N_6 := \cos^7\theta \Big[39\, a^{9/2}b^3 + 43\, a^{9/2}b^2 - 39\, a^{7/2}b^3 + 13\, a^{9/2}b - 43\, a^{7/2}b^2 + a^{9/2} - 13\, a^{7/2}b - a^{7/2} \Big]$$
$$+ \sqrt{-1}\cos^6\theta \sin\theta \Big[199\, a^4b^{7/2} + 315\, a^4b^{5/2} - 199\, a^3b^{7/2} + 141\, a^4b^{3/2} - 315\, a^3b^{5/2} + 17\, a^4b^{1/2} - 141\, a^3b^{3/2} - 17\, a^3b^{1/2} \Big]$$
$$+ \cos^5\theta \sin^2\theta \Big[-\, 419\, a^{7/2}b^4 - 919\, a^{7/2}b^3 + 419\, a^{5/2}b^4 - 577\, a^{7/2}b^2 + 919\, a^{5/2}b^3 - 101\, a^{7/2}b + 577\, a^{5/2}b^2 + 101\, a^{5/2}b \Big]$$
$$+ \sqrt{-1}\cos^4\theta \sin^3\theta \Big[-\, 467\, a^3b^{9/2} - 1399\, a^3b^{7/2} + 467\, a^2b^{9/2} - 1201\, a^3b^{5/2} + 1399\, a^2b^{7/2} - 293\, a^3b^{3/2} + 1201\, a^2b^{5/2} + 293\, a^2b^{3/2} \Big]$$
$$+ \cos^3\theta \sin^4\theta \Big[293\, a^{5/2}b^5 + 1201\, a^{5/2}b^4 - 293\, a^{3/2}b^5 + 1399\, a^{5/2}b^3 - 1201\, a^{3/2}b^4 + 467\, a^{5/2}b^2 - 1399\, a^{3/2}b^3 - 467\, a^{3/2}b^2 \Big]$$
$$+ \sqrt{-1}\cos^2\theta \sin^5\theta \Big[101\, a^2b^{11/2} + 577\, a^2b^{9/2} - 101\, ab^{11/2} + 919\, a^2b^{7/2} - 577\, ab^{9/2} + 419\, a^2b^{5/2} - 919\, ab^{7/2} - 419\, ab^{5/2} \Big]$$
$$+ \cos^1\theta \sin^6\theta \Big[-\, 17\, a^{3/2}b^6 - 141\, a^{3/2}b^5 + 17\, a^{1/2}b^6 - 315\, a^{3/2}b^4 + 141\, a^{1/2}b^5 - 199\, a^{3/2}b^3 + 315\, a^{1/2}b^4 + 199\, a^{1/2}b^3 \Big]$$
$$+ \sqrt{-1}\sin^7\theta \Big[-\, ab^{13/2} - 13\, ab^{11/2} + b^{13/2} - 43\, ab^{9/2} + 13\, b^{11/2} - 39\, ab^{7/2} + 43\, b^{9/2} + 39\, b^{7/2} \Big],$$

with numerator 7:

$$
\begin{aligned}
N_7 := \cos^7\theta \Big[&- 27\, a^{9/2} b^3 - 27\, a^{9/2} b^2 + 27\, a^{7/2} b^3 - 9\, a^{9/2} b + 27\, a^{7/2} b^2 - a^{9/2} + 9\, a^{7/2} b + a^{7/2} \Big] \\
+ \sqrt{-1} \cos^6\theta \sin\theta \Big[&- 135\, a^4 b^{7/2} - 207\, a^4 b^{5/2} + 135\, a^3 b^{7/2} - 93\, a^4 b^{3/2} + 207\, a^3 b^{5/2} - 13\, a^4 b^{1/2} + 93\, a^3 b^{3/2} + 13\, a^3 b^{1/2} \Big] \\
+ \cos^5\theta \sin^2\theta \Big[&279\, a^{7/2} b^4 + 615\, a^{7/2} b^3 - 279\, a^{5/2} b^4 + 381\, a^{7/2} b^2 - 615\, a^{5/2} b^3 + 69\, a^{7/2} b - 381\, a^{5/2} b^2 - 69\, a^{5/2} b \Big] \\
+ \sqrt{-1} \cos^4\theta \sin^3\theta \Big[&307\, a^3 b^{9/2} + 939\, a^3 b^{7/2} - 307\, a^2 b^{9/2} + 801\, a^3 b^{5/2} - 939\, a^2 b^{7/2} + 193\, a^3 b^{3/2} - 801\, a^2 b^{5/2} - 193\, a^2 b^{3/2} \Big] \\
+ \cos^3\theta \sin^4\theta \Big[&- 193\, a^{5/2} b^5 - 801\, a^{5/2} b^4 + 193\, a^{3/2} b^5 - 939\, a^{5/2} b^3 + 801\, a^{3/2} b^4 - 307\, a^{5/2} b^2 + 939\, a^{3/2} b^3 + 307\, a^{3/2} b^2 \Big] \\
+ \sqrt{-1} \cos^2\theta \sin^5\theta \Big[&- 69\, a^2 b^{11/2} - 381\, a^2 b^{9/2} + 69\, a b^{11/2} - 615\, a^2 b^{7/2} + 381\, a b^{9/2} - 279\, a^2 b^{5/2} + 615\, a b^{7/2} + 279\, a b^{5/2} \Big] \\
+ \cos^1\theta \sin^6\theta \Big[&13\, a^{3/2} b^6 + 93\, a^{3/2} b^5 - 13\, a^{1/2} b^6 + 207\, a^{3/2} b^4 - 93\, a^{1/2} b^5 + 135\, a^{3/2} b^3 - 207\, a^{1/2} b^4 - 135\, a^{1/2} b^3 \Big] \\
+ \sqrt{-1} \sin^7\theta \Big[&a b^{13/2} + 9\, a b^{11/2} - b^{13/2} + 27\, a b^{9/2} - 9\, b^{11/2} + 27\, a b^{7/2} - 27\, b^{9/2} - 27\, b^{7/2} \Big].
\end{aligned}
$$

- **End of proof of the Theorem:** The sum:

$$
\frac{1}{8} \boldsymbol{N}_1(\theta) + \frac{3}{4} \boldsymbol{N}_2(\theta) + \frac{1}{2} \boldsymbol{N}_3(\theta) + \frac{1}{8} \boldsymbol{N}_4(\theta) + \frac{15}{8} \boldsymbol{N}_5(\theta) + \frac{5}{4} \boldsymbol{N}_6(\theta) + \frac{15}{8} \boldsymbol{N}_7(\theta) \;=\; 0,
$$

is null. □

Theorem. [M. 2017] *For all* $a \geqslant 1$ *and* $b \geqslant 1$ *with* $(a,b) \neq (1,1)$*:*

$$
\dim_{\mathbb{R}} \mathsf{Umb}_{\mathsf{CR}}\big(E_{a,b}\big) \;=\; 1.
$$

□

This is a nice result but seems to contradict the computations by Ezhov, McLaughlin and Schmalz in Notices of the AMS, vol 58, no.1, indicating that the real vertices are never umbilical. However the curve described in Theorem 1.4. passes for $a = 1$ (and $b > 1$) through the "vertex" $(0, b^{-1/2})$.

Our computations from the previous section applied to the ellipsoid

$$|z_1|^2 + |z_2|^2 + \frac{a}{2}(z_1^2 + \bar{z}_1^2) + \frac{b}{2}(z_2^2 + \bar{z}_2^2) = 1$$

in $\mathbb{C}^2$ at the vertex $(0, \sqrt{\frac{2}{2+b}})$ give us the harmonic curvature

$$\begin{aligned} k_1 &= \frac{96(b+1)^4\left(-2(4b+7)a^3\right.}{(b+2)^2} \\ &\quad + \frac{\left.(1335b^2 + 3123b + 1450)\,a - 165(b-1)b\right)}{(b+2)^2} \\ &\neq 0 \\ k_2 &= 0. \end{aligned}$$

• $a = 1$ **subcase:**

Theorem. [Foo-Merker-Ta 2017] *For every real numbers* $b > 1$*, the curve parametrized by* $\theta \in \mathbb{R}$ *valued in* $\mathbb{C}^2 \cong \mathbb{R}^4$*:*

$$\gamma\colon \quad \theta \longmapsto \big(x(\theta) + \sqrt{-1}\, y(\theta),\ u(\theta) + \sqrt{-1}\, v(\theta)\big)$$

with components:

$$x(\theta) := 0, \qquad y(\theta) := 0,$$
$$u(\theta) := \frac{1}{\sqrt{b}} \sin\theta, \qquad v(\theta) := -\cos\theta,$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \mathsf{Umb}_{\mathsf{CR}}\big(\mathsf{E}_{1,b}\big) \subset \mathsf{E}_{1,b}$$

of the ellipsoid $\mathsf{E}_{1,b} \subset \mathbb{C}^2$ *of equation* $x^2 + y^2 + b\,u^2 + v^2 = 1$*.*

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